

Doppler-shifted inertial oscillations on a β -plane

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Abstract.

On the spherical earth, and in the absence of mean flow, the poleward propagation of near-inertial oscillations is restricted by the turning latitude. Mean currents, on the other hand, provide a way to increase the apparent frequency of near-inertial waves through Doppler shifting. In this note, we show that near-inertial oscillations can be advected to latitudes higher than their turning latitude. Associated with the poleward advection there is a squeezing of the meridional wavelength. We use a numerical model to verify this result. The squeezed inertial oscillations are vulnerable to nonlinear interactions, which could eventually lead to small-scale dissipation and mixing.

1. Introduction

It is well known that there is an asymmetry in the meridional propagation of near-inertial waves, since waves that propagate poleward soon reach their turning latitude and are reflected back toward the equator (Geisler and Dickinson, 1972; Anderson and Gill, 1979; Gill, 1984; Garrett, 2001). The theory is supported by observations (e.g., Fu, 1981; Chiswell, 2003; Alford, 2003). Furthermore, the equatorward propagation of near-inertial waves is important for the redistribution of the energy available for ocean mixing (Alford, 2003). Near-inertial waves can also interact with mean currents and meso-scale eddies during their propagation (Olbers, 1981; D’Asaro, 1995; Lee and Eriksen, 1997). Kunze (1985) showed that for near-inertial waves propagating in geostrophic shear, horizontally non-uniform relative vorticity has the same effect as the variation of the Coriolis parameter with latitude. As a consequence, near-inertial energy can be trapped in regions of anticyclonic relative vorticity. In addition, White (1972) found evidence from mooring data that a uniform mean current can cause a Doppler shift of the inertial frequency. Further evidence of this effect has been provided by a case study of Doppler-shifted inertial oscillations in the Norwegian Coastal Current (Orvik and Mork, 1995).

Zhai et al. (2004) recently studied the zonal advective spreading of storm-induced inertial oscillations in a model of the northwest Atlantic Ocean. The fact that inertial oscillations can be advected by a background flow raises the question of what happens if inertial oscillations are advected poleward beyond their turning latitude, where they cannot exist by themselves since these waves are strictly subinertial. Doppler shifting, on the other hand, provides a way to increase the apparent wave frequency. In this note we show that it is possible for near-inertial energy to be carried poleward due to Doppler shifting and we provide a simple theory to predict the change of shape of the inertial waves as they are advected poleward on a β -plane.

2. Analytic Model

We start from the reduced-gravity model and then extend the theory to a continuously stratified ocean. The equations governing linear wave motion on an f -plane in the presence

of a barotropic, uniform poleward mean flow are

$$u_t + Vu_y - fv = -g'\eta_x \quad (1)$$

$$v_t + Vv_y + fu = -g'\eta_y \quad (2)$$

$$\eta_t + V\eta_y + H(u_x + v_y) = 0 \quad (3)$$

where (u, v) are perturbation velocities in x and y directions respectively, V is the poleward mean current, H is the averaged upper layer depth and g' is the reduced gravity, defined as $g(\rho_2 - \rho_1)/\rho_2$, where ρ_1 and ρ_2 are upper and lower layer densities, respectively.

The divergence and vorticity equations are, respectively,

$$\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial y}\right)(u_x + v_y) - f(v_x - u_y) = -g'(\eta_{xx} + \eta_{yy}) \quad (4)$$

and

$$\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial y}\right)(v_x - u_y) + f(u_x + v_y) = 0. \quad (5)$$

Combining equations (3), (4) and (5), we get the equation for η

$$\left[\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial y}\right)^2 + f^2\right](\eta_t + V\eta_y) = \left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial y}\right)[g'H(\eta_{xx} + \eta_{yy})]. \quad (6)$$

Looking for solutions of the form

$$\eta = \eta_0 e^{i(kx + ly + mz - \omega t)} \quad (7)$$

where (k, l, m) are the wavenumbers and ω is the frequency, leads to the dispersion relationship

$$(\omega - Vl)^2 = f^2 + g'H(k^2 + l^2) \quad (8)$$

plotted in Fig. 1. We note that ω can be less than f when there is a poleward flow in the northern hemisphere ($V > 0$ and $l < 0$). The dispersion relationship can be extended to a continuously stratified ocean, and takes the form

$$(\omega - Vl)^2 = f^2 + \frac{N^2(k^2 + l^2)}{m^2} \quad (9)$$

2.1 On a β -plane

We first investigate this problem in the “diagnostic” case. “Diagnostic” means that the density field is specified and the horizontal pressure gradients are no longer interactive with the flow, so that baroclinic dispersion of inertial-gravity waves is excluded¹. In this case, the dispersion relationship reduces to

$$(\omega - Vl)^2 = (f_0 + \beta y)^2 \quad (10)$$

where f is replaced by $f_0 + \beta y$ on a β -plane. f_0 is the local inertial frequency at the latitude where the near-inertial waves are generated and β is the variation of f with latitude. For the near-inertial waves, ω is very close to f_0 , which leaves

$$-Vl = \beta y. \quad (11)$$

The meridional wavenumber is determined by $-\beta y/V$, where $y/V = t$ is the time scale. For the poleward mean current, V is positive, y is positive and β is positive, so l is negative and its amplitude increases linearly with latitude during the advection. This indicates that the inertial oscillations shrink meridionally when carried poleward (Fig. 2). The same is true for inertial oscillations that are carried equatorward.

The near-inertial energy is carried by the group velocity. The horizontal group velocity in the diagnostic case is

$$\frac{\partial \omega}{\partial l} = V \quad (12)$$

which means the near-inertial energy is transported poleward solely by the mean current at the speed of the mean current velocity and the near-inertial waves act as passive tracers. The vertical group velocity is $\partial \omega / \partial m = 0$ in this case, which indicates that the near-inertial energy is trapped in the mixed layer and dissipated there, and this energy is not available for deep ocean mixing.

However, in the real ocean, the density field is free to interact with the flow. In this case, the baroclinic dispersion can play a role and the near-inertial waves become active

¹The diagnostic case is appropriate when the horizontal length scales are large compared to the internal radius of deformation

tracers,

$$(\omega - Vl)^2 = (f_0 + \beta y)^2 + \frac{N^2 l^2}{m^2} \quad (13)$$

where only vertical and meridional propagation are considered. Assuming $\omega \approx f_0$ yields $l < -\beta y/V$, which indicates that the inertial oscillations shrink quicker meridionally than in the diagnostic case. The horizontal group velocity is expressed in the prognostic case as

$$\frac{\partial \omega}{\partial l} = V + \frac{N^2 l}{m^2(\omega - Vl)} \quad (14)$$

where the horizontal group velocity is determined by the summation of the mean current velocity and the modified horizontal wave dispersion. Since l is negative, the near-inertial energy is transported poleward at a speed less than the mean current velocity. The vertical group velocity is expressed in the prognostic case as

$$\frac{\partial \omega}{\partial m} = -\frac{N^2 l^2}{m^3(\omega - Vl)}. \quad (15)$$

The negative sign indicates the downward propagation of the near-inertial energy. Since $\omega - Vl > \omega$, the amplitude of the vertical group velocity is reduced by a factor of $(\omega - Vl)/\omega$. When the near-inertial waves are carried poleward by the mean current, the downward propagation of the near-inertial energy is reduced and less energy escapes from the mixed layer to the deep ocean.

Zhai et al. (2004) showed that inertial energy can be carried by a background current (in their case, the Gulf Stream) to remote regions in a model of the northwest Atlantic Ocean. The dominance of advection over wave dispersion can be easily demonstrated here. The zonal horizontal group velocity in the presence of a zonal mean flow is

$$\frac{\partial \omega}{\partial k} = U + \frac{N^2 k}{m^2(\omega - Vk)} \quad (16)$$

where U is the zonal mean flow. For the size of their storm, k is about $2 \times 10^{-5} \text{ m}^{-1}$, ω is about 10^{-4} s^{-1} , we take $N^2/m^2 = 1.0 \text{ m}^2 \text{ s}^{-2}$ and the background current velocity U is close to 1 m s^{-1} . Thus

$$\frac{\text{dispersive processes}}{\text{advective processes}} = \frac{N^2 k}{m^2(\omega - Vk)U} = \frac{0.2}{1} \quad (17)$$

which indicates that advective processes dominate the near-inertial wave dispersion in their case.

2.2 On an f -plane

On the f -plane, the β -effect is excluded. In the diagnostic case, the dispersion relationship reduces to

$$(\omega - Vl)^2 = f_0^2 \quad (18)$$

which shows that $Vl = \text{constant}$. As long as the poleward background current V is spatially uniform, the meridional wavenumber l is constant, which indicates that the inertial oscillations keep their shape during the poleward advection.

In the prognostic case, baroclinic dispersion can play a role. The northward dispersion is enhanced by the advection, while the southward dispersion is reduced as seen from the dispersion curve (Fig. 2). Depending on the strength of the mean flow, the first baroclinic mode is the mode that can most easily overcome the poleward advection and propagate equatorward.

3. Numerical Model

The ocean model used here is the same as in Zhai et al. (2004), except that we use an idealized rectangular domain. The model covers the area between 30°W and 60°W and between 30°N and 60°N, with two open boundaries at the south and the north and two solid boundaries at the east and the west. The horizontal resolution is about 20 km and there are 31 unevenly distributed vertical levels with the centers of the top five levels located at 5, 16, 29, 44 and 61 m, respectively. The stratification is horizontally uniform, with a vertical temperature structure representative of the mid-latitude Atlantic Ocean (Fig. 3). The salinity is set everywhere uniform. An initial poleward current of 50 cm s^{-1} is introduced everywhere in the domain and maintained by the open boundaries throughout the simulation. The bottom relief is designed in such a way as to compensate the variation of the Coriolis parameter with latitude. The water depth is a function of longitude and latitude and is given by

$$H(x, y) = H(x) \times 2\sin(\phi) \quad (19)$$

where the zonal dependence $H(x)$ is a linear slope and ϕ is the latitude. In this way,

$$\frac{f}{H} = \frac{2\Omega\sin(\phi)}{H(x) \times 2\sin(\phi)} = \frac{\Omega}{H(x)} \quad (20)$$

where Ω is the Earth's rotation rate. The poleward flow is maintained and almost spatially uniform, following the f/H contours rather than forming an intensified western boundary current as happens with a flat bottom. Storm forcing is specified following Chang and Anthes (1978) and used to generate the inertial oscillations. The wind stress for the storm is

$$\tau = \tau_{max} \times \begin{cases} r/r_{min} & 0 \leq r \leq r_{min} \\ (r_{max} - r)/(r_{max} - r_{min}) & r_{min} \leq r \leq r_{max} \\ 0 & r \geq r_{max} \end{cases} \quad (21)$$

where τ is the amplitude of the tangential wind stress with respect to the storm center, and r is the radial distance from the center. Here, we put $r_{min} = 30$ km, $r_{max} = 300$ km, and $\tau_{max} = 3$ N m⁻² for a typical storm, the same as that in Zhai et al. (2004). The storm track is specified to be zonal from 35°W and 55°W at 43°N latitude and the translation speed of the storm is about 8.5 m s⁻¹. Radiation open boundary conditions are used at the south and the north boundaries. Since the boundaries are far away from the area we are interested in, they are small in their effects.

4. Results

4.1 On a β -plane

Two prognostic model runs are conducted on a β -plane, one with the storm forcing and the other without the storm forcing. The velocity differences between the two model runs are used to represent the oceanic response to the storm forcing. In order to extract the near-inertial response, a bandpass filter centered at the local (43°N) inertial frequency is used. The temporal and spatial evolution of the inertial-band filtered zonal currents at the sea surface is shown in Fig. 4. The near-inertial currents are initially biased to the right of the storm track (not shown) consistent with previous studies (e.g., Price, 1981). They are gradually advected poleward and the inertial oscillations are almost centered at the storm track at day 4. As the inertial oscillations are carried further poleward, they are squeezed meridionally as predicted by the linear theory, while the zonal wavenumber is well preserved. The β -dispersion effect is also evident after day 7, indicated by the near-inertial waves propagating equatorward, but it seems that most energy is carried

poleward of the storm track. A vertical transect along the middle longitude is shown in Fig. 5. This figure is very similar to Fig. 12 in Gill (1984), but note that the source of these equatorward-propagating waves is carried several hundred kilometers poleward of the storm track by the background flow. It seems only the first mode of the baroclinic inertial-gravity waves can make its way equatorward, as indicated by the 180° phase difference between the near-surface and near-bottom currents.

Two additional diagnostic model runs (i.e., one with the storm forcing and one without) are conducted on the β -plane, with the density field specified from the initial condition, in which case the horizontal pressure gradients are independent of the model-calculated temperature so that the baroclinic dispersion of the inertial-gravity waves is excluded. This effect is evident in Fig. 6. There is no baroclinic dispersion and the near-inertial energy is confined in the mixed layer except for the (deep) inertial pumping which is also carried poleward of the storm track. The inertial oscillations act like passive tracers in the diagnostic case and the near-inertial energy occurs only poleward of the storm track at day 13, and is eventually dissipated there. For $t \approx 10$ days and $V \approx 50 \text{ m s}^{-1}$, the advection distance is roughly about 430 km, which is consistent with what is shown in Fig. 7. The meridional width of the inertial oscillations is about 300 km at day 13 after being advected poleward for about 500 km, close to the analytical prediction in Fig. 2. The advection distance of the inertial oscillations in the prognostic run is a little shorter than that in the diagnostic run, due to the second term in equation (14), which is negative and representing the wave propagation. The most revealing fact is that the diagnostic run captures the essential features of the prognostic run, i.e., the poleward advection of the near-inertial energy and the meridional squeezing of the near-inertial oscillations (compare Fig. 4 with Fig. 7).

4.2 On an f -plane

In the diagnostic run on the f -plane, the inertial oscillations act solely as passive tracers (Fig. 8). There is no squeezing of the meridional wavelength and the shape of the inertial oscillations are well preserved during the advection, which is consistent with the analytical solution. In the prognostic case on the f -plane, baroclinic dispersion takes

effect and there is energy leakage both northward and southward through the propagation of the near-inertial waves (Fig. 9). This can be explained by the concepts of modal separation and modal interference as described in Gill (1984) and Zervakis and Levine (1995). The inertial oscillations, though carried northward by the mean current, do not change much in their shape in contrast to what happens on the β -plane.

5. Summary and Discussion

Inertial oscillations can be carried poleward by a mean flow beyond their turning latitude due to the Doppler shift effect. The inertial oscillations shrink meridionally with latitude during this advection. As the scales become smaller, the near-inertial waves are more vulnerable to nonlinear interactions, which could eventually lead to small-scale dissipation and mixing. This advection-induced mixing occurs poleward of their source regions. Since a given energy level at higher latitudes causes much more mixing than at lower latitudes (Gregg et al., 2003; Garrett, 2003), a mechanism for transporting inertial energy to higher latitudes could be important for understanding mixing in the ocean. The phenomenon discussed in this paper could be applied to the North Atlantic Current, the Norwegian Coastal Current, and other poleward currents, even though those are more complicated environments and subject to additional physics.

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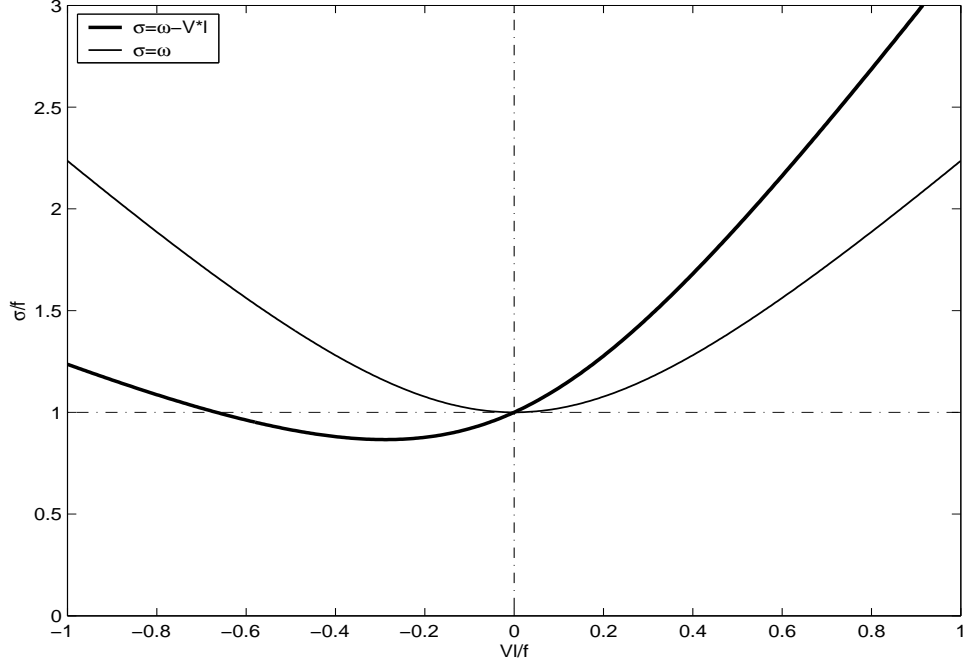


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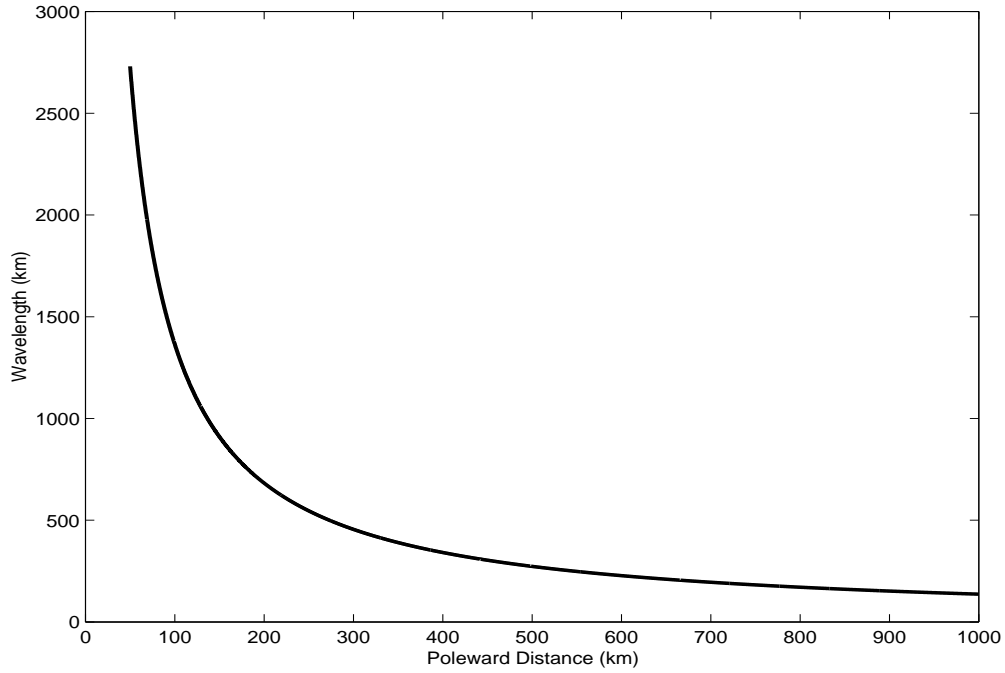


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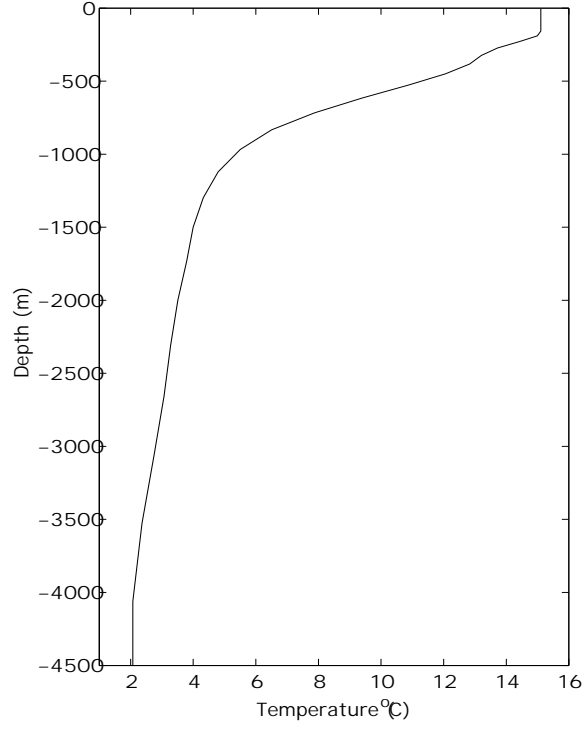


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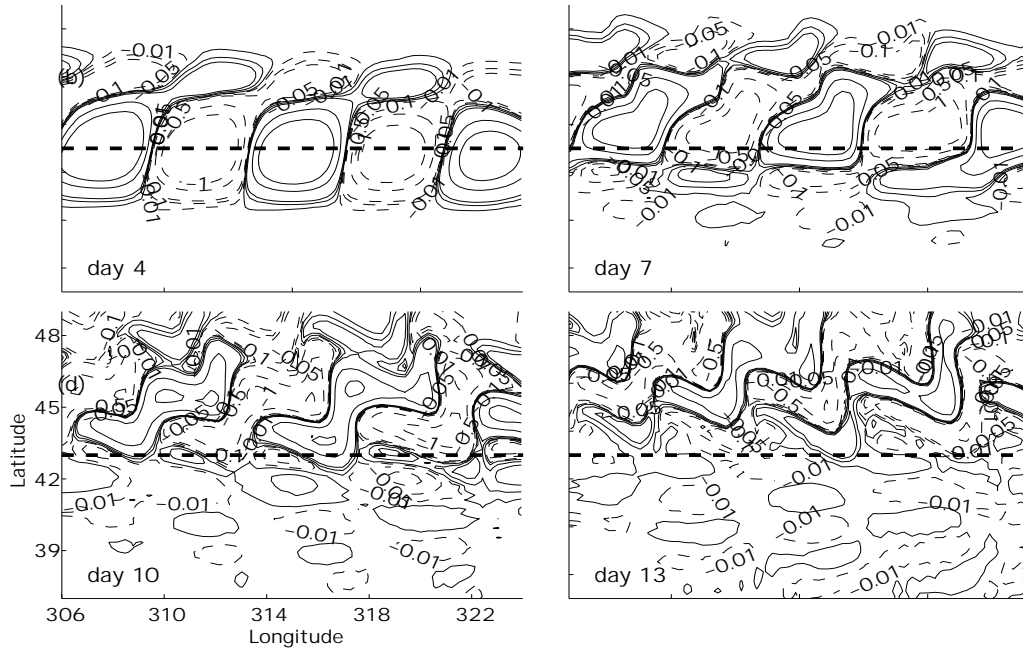


Fig. 4: Temporal evolution of the inertial-band filtered zonal current at the sea surface in the prognostic run on a β -plane (unit: $\text{m}^2 \text{s}^{-2}$). The dashed line represents the storm track.

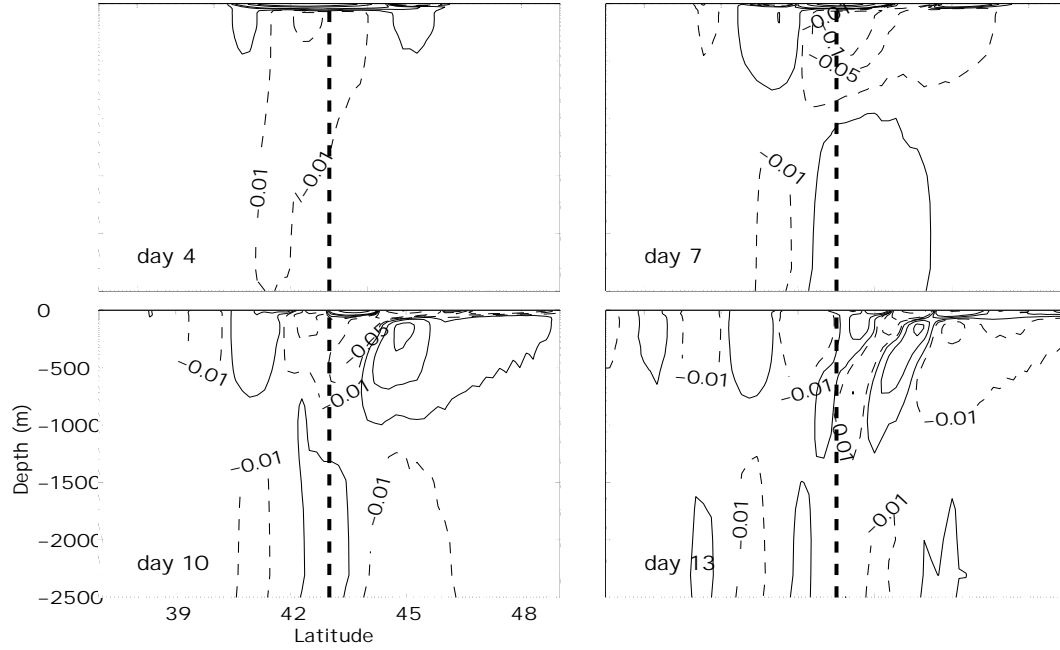


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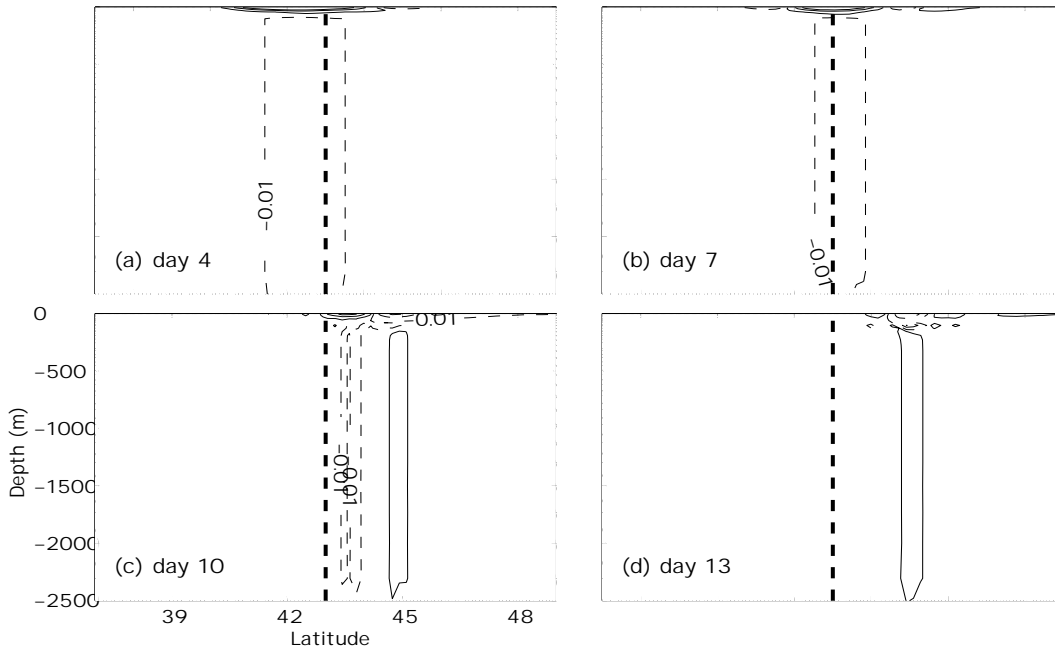


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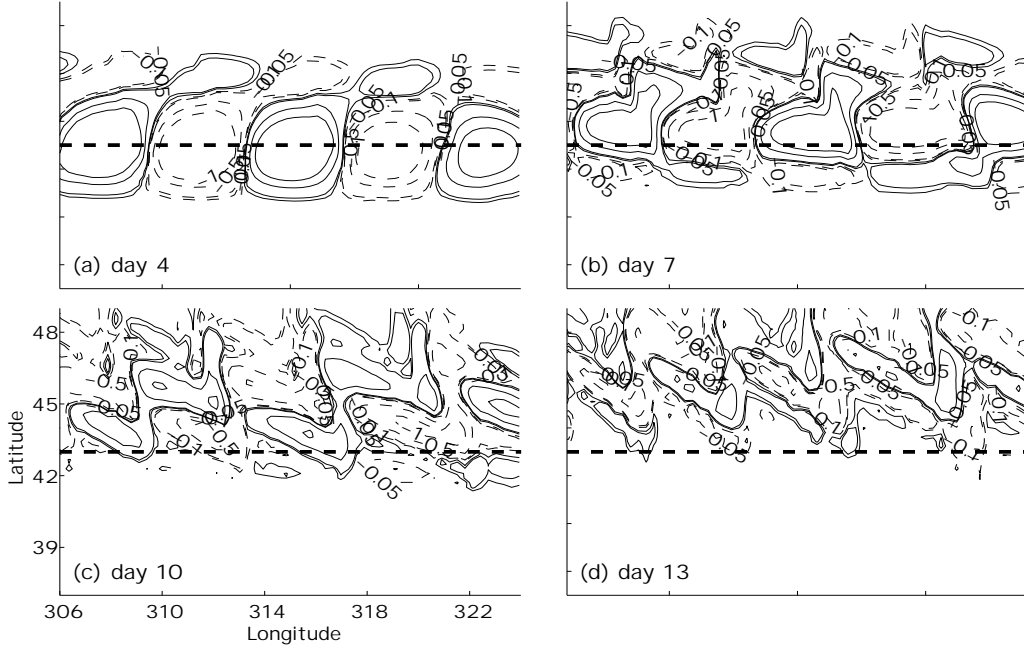


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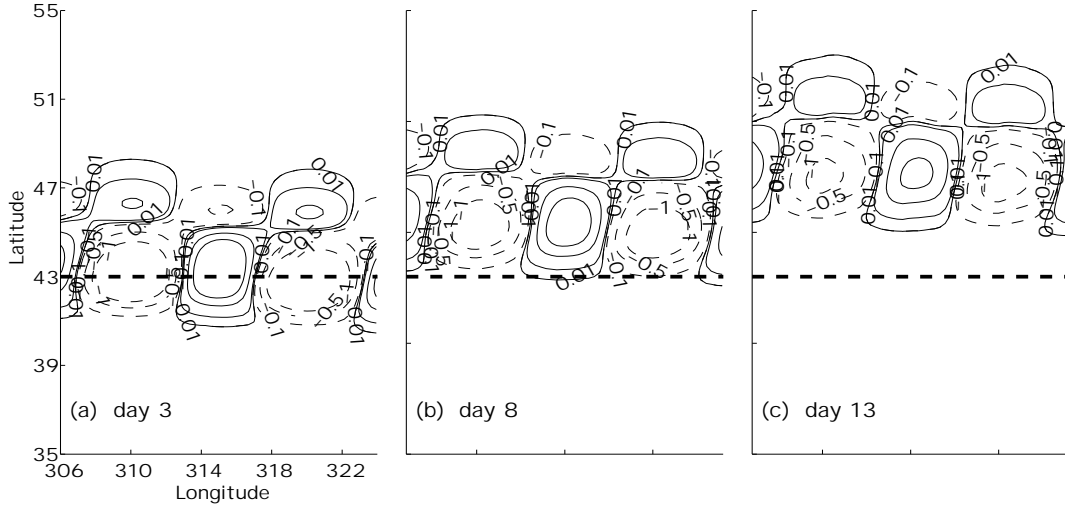


Fig. 8: Temporal evolution of the zonal current at the sea surface in the diagnostic run on a f -plane (unit: $m^2 s^{-2}$). The dashed line represents the storm track.

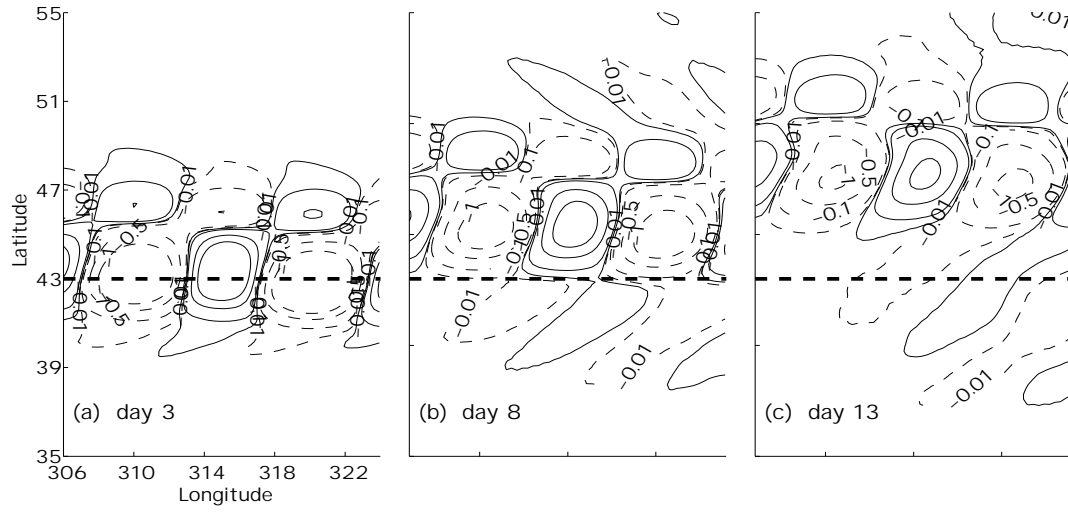


Fig. 9: Temporal evolution of the zonal current at the sea surface in the prognostic run on a f -plane (unit: $m^2 s^{-2}$). The dashed line represents the storm track.