Interpreting eddy fluxes

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All figures are available in electronic format.

Abstract

New ways of interpreting eddy tracer fluxes averaged at constant height are presented and compared with previous interpretations, in particular the Transformed Eulerian Mean (TEM) of Andrews and McIntyre, and the Temporal Residual Mean (TRM) of McDougall and McIntosh. The previous decompositions lead to (slightly different) residual streamfunctions, but also imply eddy–induced diapycnal mixing that may not be physically justified. For example, the implied diapycnal mixing is not guaranteed to vanish for adiabatic flow and can become large (compared to the mean diabatic forcing) for weakly diabatic flow as revealed in numerical simulations.

We introduce a new, "adiabatic TEM" (TEM-A) that leads to a residual streamfunction with zero eddy–induced diapycnal diffusion by construction, even when the instantaneous diabatic forcing is non-zero. This is achieved by subtracting an optimal rotational component from the eddy tracer fluxes. This rotational flux is found from the solution of a linear minimization problem. A variant of the method yields the "generalized TRM" (TRM-G) with eddy–induced (or enhanced) diapycnal diffusion in the steady case only if there is a correlation between perturbations in the diabatic forcing and the tracer. It follows that for steady, adiabatic flow, the eddy-induced diapycnal diffusivity is zero for both the TEM-A and TRM-G cases. In the numerical simulations the eddy–induced diapycnal diffusivities in TRM-G are orders of magnitude larger than the mean diabatic forcing, but it is shown that in the steady, adiabatic limit TEM-A and TRM-G converge.

1 Introduction

The Boussinesq form of the conservation equation for a tracer with concentration b in the ocean (or the atmosphere) is given by

$$\frac{\partial b}{\partial t} + \nabla \cdot (\mathbf{u} b) = Q \tag{1}$$

where $\mathbf{u}$ denotes the instantaneous, three-dimensional velocity and Q a forcing term. However, both the ocean and atmosphere are turbulent fluids, full of "rapidly evolving perturbations" (eddies) on a "slowly evolving mean state". The presence of the eddies means that the instantaneous tracer distribution is often of little interest; instead, it is the dynamics and evolution of an "averaged" state which is important. This, in turn, requires the definition of an average or filter with which to view the dynamics and evolution of the tracer field, and, in turn, determine what is meant by "rapid","slow" and "mean state".

A simple example is the zonal mean; that is b, $\mathbf{u}$ and Q are decomposed into zonal averages at constant height and deviations from that average.

$$\bar{b} = (x_2 - x_1)^{-1} \int_{x_1}^{x_2} b \, dx \quad , \quad b' = b - \bar{b} \tag{2}$$

and analogously for $\mathbf{u}$ and Q. Substituting the decomposition given by Eq. 2 into the instantaneous tracer budget and averaging the result leads to the equation for the mean tracer $\bar{b}$ given by

$$\frac{\partial \bar{b}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}} \bar{b}) + \nabla \cdot (\overline{\mathbf{u}' b'}) \quad = \quad \bar{Q} \tag{3}$$

An immediate difficulty is presented by the "eddy tracer fluxes" given[1] by $\overline{\mathbf{u}' b'}$. These fluxes couple the mean tracer budget to that of the perturbations, such that the evolution of the perturbations has to be known to understand the mean tracer equation. Of course, the solution to this problem is thought to be given by "parameterizing" the perturbation quantities in terms of the mean quantities. However, before parameterizing the effect of the eddy tracer fluxes, it is necessary to understand and interpret them. Understanding and interpreting the eddy fluxes is the focus of the present paper.

Some insight into the nature of the eddy tracer fluxes can be obtained by considering a layered framework, in which instantaneous contours of b are taken as layer interfaces. In

[1] Note that the vectors $\bar{\mathbf{u}}$, $\mathbf{u}'$, the operator ∇ and correspondingly the fluxes $\bar{\mathbf{u}}\bar{b}$ and $\overline{\mathbf{u}'b'}$ in Eq. 2 and in the following section are two dimensional, that is their zonal component vanishes.

the continuous limit of infinitesimal layer thickness, this is the same as using b as the vertical coordinate[2]. Taking b to be potential density[3] then corresponds to using "isopycnal coordinates" (e.g. McDougall (1987)). Since the interior oceanic flow is almost adiabatic (Wüst, 1935), the diabatic forcing Q is expected to be small in the ocean interior and to be associated with weak diapycnal mixing. In the limit of vanishing instantaneous forcing Q, there is no instantaneous exchange of b across layers (isopycnals) and it is easy to see that in the isopycnally-averaged budget for b there can be no cross-isopycnal flux. Indeed, averaging in isopycnal coordinates the eddy flux covariance $\overline{\mathbf{u}'b'}$ is zero by construction and the only forcing for the averaged equation is the mean forcing (related to Q) controlling the amount of cross-isopycnal flux. Based on these considerations, we can formulate the following statement:

i) If there is no instantaneous diabatic forcing, Q, there should be no diabatic forcing in the mean budget for b. As a consequence, the effect of the eddy tracer fluxes must be entirely expressed as an advective flux of the mean tracer.

However, going back to z–coordinates, the above property of the eddy fluxes is far from obvious. In fact, eddy tracer fluxes averaged at constant height (instead of at constant b) usually show strong diapycnal components even for the steady, weakly diabatic case, suggesting artificially strong eddy-induced diapycnal mixing as we shall show below. On the other hand, z–coordinates are convenient and simple to use for both analytical considerations and numerical calculations. Therefore, an overwhelming number of analytical and numerical models are based on z–coordinates, instead of layered or isopycnal coordinates. It is therefore regarded as a benefit to carry over the character of eddy tracer fluxes as revealed above in the layered framework to the mean tracer budget in z–coordinates. This is the aim of the present study.

The statement i) above is a feature of the eddy tracer flux parameterization of Gent et al. (1995) (the so-called Gent and McWilliams (GM) parameterization). In GM, the eddy fluxes are assumed to be adiabatic in their effect, and consequently, the parameterization is written entirely in terms of an advective velocity. Although this is certainly a reasonable first order approximation, Radko and Marshall (2004) have raised the possibility that eddy-induced diapycnal mixing, especially in association with the surface and bottom boundary layers (where diabatic effects are relatively strong) might be important in the dynamics of the large-scale

[2]Assuming that b is a monotonic function of depth. See Nakamura (2001) and Nurser and Lee (2004) for a generalization of this approach for non–monotonic functional forms of b.

[3]Note that we assume for simplicity an equation of state in which potential density and neutral density are the same.

ocean circulation. In particular, their work implies that eddy fluxes do indeed have a diabatic component, with the implication that a parameterization of the eddy tracer flux should include a diabatic component as well. Tandon and Garrett (1996) had earlier pointed out that the release of available potential energy from the mean state that is implied by the GM parameterization must, in turn, be dissipated in some way. These authors argue that the dissipation implies significant diapycnal mixing in association with eddies, and they point out the likely importance of the surface and bottom boundary layers, especially air-sea interaction at the surface, in accounting for this dissipation.

The issue of diabatic eddy transfer is one we discuss only in passing in the present paper. Rather, we concentrate on the adiabatic interpretation of eddy fluxes, and leave a comprehensive treatment of diabatic effects to a later paper. In particular, we show that it is always possible to decompose the eddy tracer flux into advective and rotational components so that the eddy tracer flux appears in the mean tracer equation as being entirely adiabatic, even if the diabatic forcing is non-zero. The result is a generalization of the Transformed Eulerian Mean (TEM) of Andrews and McIntyre (1976) to a fully adiabatic formulation. The theory naturally leads to an extension of the Temporal Residual Mean (TRM) of McDougall and McIntosh (1996). In this generalized TRM (TRM-G), diabatic transfer in the mean is achieved by an eddy diffusivity that is determined either by changes in the local eddy variance with time or by the local, irreversible dissipation of eddy variance, or both. In a statistically steady state, both the new adiabatic TEM (TEM-A) and the generalized TRM (TRM-G) are consistent with statement (i) above; namely, if there is no instantaneous diabatic forcing, then there is also no diabatic transfer implied by the mean equations when averaged in z-coordinates. The TEM-A is also consistent with statement (i) in unsteady situations.

We begin in Section 2 by reviewing the eddy tracer flux decompositions of Andrews and McIntyre (1976, 1978) (that is, the Transformed Eulerian Mean (TEM) summarized in Andrews et al. (1987)). We also discuss the extensions to the TEM introduced by McDougall and McIntosh (1996), Held and Schneider (1999) and Medvedev and Greatbatch (2003), and complement these approaches with the two new ways of interpreting eddy tracer fluxes (the TEM-A and the TRM-G mentioned above). In Section 3, we illustrate the different methods of decomposing eddy fluxes using idealized numerical experiments, and the last section summarizes the results. We have added several appendices to the main text, in which we outline the consequences of temporal (instead of zonal) averaging for the new ways of interpreting eddy tracer fluxes, and clarify some aspects of the relationship between the new methods and previous approaches.

2 The eddy fluxes

In this section we review several ways of interpreting the eddy tracer flux resulting from zonal averaging at constant height and complement them with the new approaches for the interpretation of eddy tracer flux. Note that all results carry over to temporal averaging in three dimensions, as outlined in Appendix A. Note also that in the sections concerning the zonal mean problem we use the following notation (Hasselmann, 1982)

$$\nabla\alpha = \begin{pmatrix} \alpha_y \\ \alpha_z \end{pmatrix} \quad \text{and} \quad \underset{\neg}{\nabla}\alpha = \begin{pmatrix} -\alpha_z \\ \alpha_y \end{pmatrix} \tag{4}$$

where the subscripts y and z denote differentiation of the scalar α in the meridional and vertical direction, respectively, and the vector subscript $\neg$ anti–clockwise rotation by 90^o.

The Transformed Eulerian Mean (TEM-I)

The Transformed Eulerian Mean method of Andrews and McIntyre (1976) (TEM-I hereafter) introduces a decomposition of the eddy tracer flux $\mathbf{F}$ into a part aligned along contours of $\bar{b}$ and a correction term

$$\mathbf{F} = \begin{pmatrix} \overline{v'b'} \\ \overline{w'b'} \end{pmatrix} = B\underset{\neg}{\nabla}\bar{b} + \delta\mathbf{e}^{(z)}. \tag{5}$$

Here, the ad hoc choice $B = -\overline{v'b'}/\bar{b}_z$ yields a streamfunction for an eddy–induced tracer advection velocity $\mathbf{u}_{eddy} = -\underset{\neg}{\nabla}B$. However, the representation of the advective nature of $\mathbf{F}$ is only exact with the remainder $\delta\mathbf{e}^{(z)}$, with $\delta = \overline{v'b'}/\bar{b}_z\bar{b}_y + \overline{w'b'}$. Substituting this expression for $\mathbf{F}$ into the mean tracer budget (Eq. 3) yields

$$\bar{b}_t + (\bar{\mathbf{u}} + \mathbf{u}_{eddy}) \cdot \nabla\bar{b} \quad = \quad \bar{Q} - \delta_z \tag{6}$$

In general, the remainder $\delta = \overline{v'b'}\bar{b}_y/\bar{b}_z + \overline{w'b'}$ is not zero; only if the eddy flux $\mathbf{F}$ is entirely along contours of $\bar{b}$ does the remainder δ vanish. Taking b to be density and assuming a strongly stratified situation ($\bar{b}_y << \bar{b}_z$ and accordingly $\overline{w'b'} << \overline{v'b'}$) the remainder δ can become small. In this limit (or for $\mathbf{F} \cdot \nabla\bar{b} = 0$), the effect of $\nabla \cdot \mathbf{F}$ can be represented by the eddy–induced tracer advection velocity $-\underset{\neg}{\nabla}B = \mathbf{u}_{eddy}$, which together with the mean velocity $\bar{\mathbf{u}}$ advects the tracer $\bar{b}$. The sum $\mathbf{u}_{eddy} + \bar{\mathbf{u}}$ is sometimes called the “residual velocity”. However, we shall show below with numerical simulations that the remainder δ_z is not small in general, leading to substantial deviations in the behavior of the residual streamfunction in the TEM-I from the ideal case implied by the layered framework. On the other hand, we shall also show below that it is possible to define a streamfunction with a different, decomposition of $\mathbf{F}$, which is in agreement with our knowledge from the layered framework.

An alternative TEM (TEM-II)

The alternative TEM flux decomposition of Held and Schneider (1999) (TEM-II hereafter) uses the vertical eddy fluxes to construct a streamfunction for the eddy–induced velocity:

$$\mathbf{F} = B \underset{\neg}{\nabla} \bar{b} + \delta \mathbf{e}^{(y)} \quad , \quad \bar{b}_t + (\bar{\mathbf{u}} - \underset{\neg}{\nabla} B) \cdot \nabla \bar{b} \;=\; \bar{Q} - \delta_y \tag{7}$$

where $B = \overline{w'b'}/\bar{b}_y$ and $\delta = \overline{w'b'}/\bar{b}_y \bar{b}_z + \overline{v'b'}$. The remainder δ is again zero if the eddy flux has no component across contours of $\bar{b}$. In contrast to the TEM-I, the TEM-II decomposition can become a good approximation for situations in which $\bar{b}_y >> \bar{b}_z$, e. g. in the oceanic mixed layer, since then the remainder δ can become small.

A generalized TEM (TEM-G)

It is possible and more elegant to combine the TEM-I and TEM-II in the following way. We decompose the eddy tracer flux in directions along and across contours of $\bar{b}$

$$\mathbf{F} \;=\; B \underset{\neg}{\nabla} \bar{b} + K \nabla \bar{b} \tag{8}$$

where B and K are given by projections of the eddy flux along and across contours of $\bar{b}$ respectively

$$B \;=\; |\nabla \bar{b}|^{-2} \mathbf{F} \cdot \underset{\neg}{\nabla} \bar{b} \quad , \quad K = |\nabla \bar{b}|^{-2} \mathbf{F} \cdot \nabla \bar{b} \tag{9}$$

Such a decomposition (TEM-G hereafter) can be found in Andrews and McIntyre (1978) and is used by Nakamura (2001), Greatbatch (2001) and Olbers and Visbeck (2003). Note that B denotes again a streamfunction for the eddy–induced advection velocity. In addition, an eddy–induced diffusion term with diffusivity K shows up in the mean tracer budget

$$\bar{b}_t + (\bar{\mathbf{u}} - \underset{\neg}{\nabla} B) \cdot \nabla \bar{b} \;=\; \bar{Q} - \nabla \cdot K \nabla \bar{b} \tag{10}$$

The eddy–induced diapycnal diffusivity K vanishes for $\mathbf{F} \cdot \nabla \bar{b} = 0$ (as for the remainder δ in the TEM-I and TEM-II cases). In the limits $\bar{b}_y >> \bar{b}_z$ or $\bar{b}_y << \bar{b}_z$ the streamfunction

B equals that in the TEM-II or TEM-I cases, respectively. Likewise, in these limits, the eddy–induced diapycnal diffusivity K becomes equal to $\delta/\bar{b}_y$ or $\delta/\bar{b}_z$, respectively. This feature makes the TEM-G flux decomposition useful for the more general situation and especially for the transition regions between the ocean interior and the surface and bottom boundary layers.

We shall use and diagnose TEM-G in the numerical experiments discussed below instead of the TEM-I or TEM-II flux decompositions, since the form of a diapycnal diffusivity K, rather than the remainder terms δ, offers a convenient way to quantify the effect of the remainder in comparison with explicit diapycnal diffusivities specified in the numerical experiments, and our knowledge of diapycnal diffusivities in the ocean.

The adiabatic Transformed Eulerian Mean (TEM-A)

We proceed now by aiming to minimize the remainder (δ or K) due to the non–advective character of the eddy tracer fluxes in the mean tracer budget. In fact, we shall show that this remainder can be set to zero. Consider the flux decomposition

$$\mathbf{F} \;=\; \underset{\neg}{\nabla}\theta + B\underset{\neg}{\nabla}\bar{b} + K\nabla\bar{b} \tag{11}$$

where $\underset{\neg}{\nabla}\theta$ serves as a rotational gauge flux which drops out when taking the divergence of $\mathbf{F}$. The eddy streamfunction B and diapycnal diffusivity K are now given by

$$B \;=\; |\nabla\bar{b}|^{-2}(\mathbf{F} - \underset{\neg}{\nabla}\theta)\cdot\underset{\neg}{\nabla}\bar{b} \quad , \quad K = |\nabla\bar{b}|^{-2}(\mathbf{F} - \underset{\neg}{\nabla}\theta)\cdot\nabla\bar{b} \tag{12}$$

Since $\underset{\neg}{\nabla}\theta$ makes no contribution to the mean tracer budget, we have flexibility in the choice of θ, a flexibility we now exploit. In particular, we can ask for a θ that minimizes (in some sense) $K\nabla\bar{b}$ (that is the cross–isopycnal eddy flux, when specifying b as potential density), leading to the following minimization problem

$$\frac{1}{2}\int W(K\nabla\bar{b})^2 dydz = min \tag{13}$$

where $W(y,z)$ denotes a weighting function which is specified below. The corresponding Euler–Lagrange equation is

$$\underset{\neg}{\nabla}\bar{b}\cdot\nabla\left[\frac{W}{|\nabla\bar{b}|^2}\left(\mathbf{F}\cdot\nabla\bar{b} + \nabla\theta\cdot\underset{\neg}{\nabla}\bar{b}\right)\right] = 0 \tag{14}$$

which states that the term inside the brackets is constant along contours of $\bar{b}$. By specifying W as $W = |\nabla\bar{b}|$, it is convenient to use new coordinates along (s) and perpendicular (n) to contours of $\bar{b}$ for the bracketed term so that

$$\mathbf{F} \cdot \frac{\nabla \bar{b}}{|\nabla \bar{b}|} + \nabla\theta \cdot \frac{\underset{\neg}{\nabla} \bar{b}}{|\nabla \bar{b}|} = G(\bar{b}) \quad \textit{transforms to} \quad F_{\perp}(s,n) + \frac{\partial \theta(s,n)}{\partial s} = G(n) \tag{15}$$

where $F_{\perp} = \mathbf{F} \cdot \frac{\nabla \bar{b}}{|\nabla \bar{b}|}$ denotes the component of the eddy tracer flux across contours of $\bar{b}$. Without loss of generality[4] we can set $G = 0$ and

$$\theta(s,n) = -\int_{s_0}^{s} F_{\perp} ds' \quad \text{and} \quad \theta(s_0, n) = 0 \tag{16}$$

and we get, in turn,

$$K = |\nabla \bar{b}|^{-1} (F_{\perp} + \frac{\partial \theta}{\partial s}) = 0; \tag{17}$$

that is, zero eddy–induced diffusivity K as the minimal value.

Eq. 16 states that the optimal θ is given by the integral along a $\bar{b}$ contour of the eddy tracer flux component across that contour of $\bar{b}$. Subtracting the gauge flux $\underset{\neg}{\nabla}\theta$ from the eddy tracer flux gives zero eddy–induced diffusivity K. This is the main point of the paper. It should be noted that θ, as defined by Eq. 16, is unique up to the choice of where to start the integration on the contour of $\bar{b}$, i. e. $s = s_0$. Choosing a different s_0 changes θ and, in turn, B by a function of $\bar{b}$ (but has no effect on K), implying that the eddy–induced flow can be changed by a free component along $\bar{b}$ contours. Note also that with this definition for θ, the rotational part of the eddy flux, and in consequence the definition of the eddy streamfunction, becomes non–local.

Using this eddy tracer flux decomposition in the mean tracer budget yields

$$\bar{b}_t + (\bar{\mathbf{u}} - \underset{\neg}{\nabla} B) \cdot \nabla \bar{b} \;\; = \;\; \bar{Q} \tag{18}$$

For $\bar{b}_t = \bar{Q} = 0$, the residual flow is purely along contours of $\bar{b}$, and, in general (including cases with time-dependence), only a non–zero $\bar{Q}$ can force cross–isopycnal residual flow. We stress that these features are the essentials of what we know from the layered framework; that is, in agreement with statement i) from section 1.

The TEM-A approaches the previous versions in the following limit: If the eddy tracer flux is entirely along contours of $\bar{b}$, the rotational potential θ and in consequence the rotational eddy

[4] The following (mathematical) caveat should be noted: In the case of a closed contour of $\bar{b}$, we obviously cannot set the constant G in Eq. 15 to zero. Here, it is not possible to set θ such that $K = 0$, instead we have to set $G(n) = \oint F_{\perp}(s,n) ds$ and get $K = |\nabla \bar{b}|^{-1} \oint F_{\perp} ds$. The minimal diffusivity K in the case of closed contours of $\bar{b}$ is given by the averaged and weighted eddy tracer flux across the respective closed contour of $\bar{b}$. However, note that this case cannot apply in the case of isopycnal coordinates for which closed isopycnals are not permitted.

tracer flux is zero. For this case the eddy streamfunction and diffusivity (which is also zero) become the same as in the TEM-G, and in the limits $\bar{b}_z << \bar{b}_y$ or $\bar{b}_z >> \bar{b}_y$ the same as in the TEM-II and TEM-I respectively.

The Temporal Residual Mean (TRM) vs. TEM-A

Adding a rotational non–divergent part to the eddy tracer fluxes $\mathbf{F}$ does not effect the mean tracer budget. However, it does effect the eddy variance equation, offering an opportunity to interpret this part of the flux. This fact was used by McDougall and McIntosh (1996) to develop the Temporal Residual Mean (TRM-I hereafter) extension to the TEM theory and later the TRM–II version (McDougall and McIntosh, 2001). We discuss here the more general and – in this context – simpler derivation, similar to what can be found in Greatbatch (2001) and Medvedev and Greatbatch (2003), and compare it with the above presented TEM-A.

The eddy tracer fluxes are again expressed as

$$\mathbf{F} \;=\; \underset{\neg}{\nabla}\theta + B\underset{\neg}{\nabla}\bar{b} + K\nabla\bar{b} \tag{19}$$

where $\underset{\neg}{\nabla}\theta$ serves as a gauge flux which drops out when taking the divergence of $\mathbf{F}$ and so does not contribute to the mean tracer budget. Motivation for a non-zero rotational potential θ comes from the budget of tracer variance ($\overline{\phi} = \overline{b'^2}/2$) since here the gauge flux shows up. The eddy variance equation is given by

$$\overline{\phi}_t + \nabla\cdot\overline{\mathbf{u}\phi} \;=\; \overline{b'Q'} - \underset{\neg}{\nabla}\theta\cdot\nabla\bar{b} - K|\nabla\bar{b}|^2 \tag{20}$$

in which the decomposition for $\mathbf{F}$ is used. Note that we could set $K = 0$ by using $\theta(s,n) = -\int_{s_0}^{s} F_\perp ds$ as before, but, for now, we hesitate to do so.

We now decompose the total (i. e. mean plus eddy) flux of eddy tracer variance into components along and across contours of $\bar{b}$ plus a rotational part, as we did before for the eddy tracer flux $\mathbf{F}$,

$$\overline{\mathbf{u}\phi} \;=\; \underset{\neg}{\nabla}\theta_2 + B_2\underset{\neg}{\nabla}\bar{b} + K_2\nabla\bar{b} \tag{21}$$

Note that we can set $K_2 = 0$ by the choice $\theta_2 = -\int_{s_0}^{s}[\overline{\mathbf{u}\phi}]_\perp ds$, but again we hesitate to do so. Instead we want to monitor the eddy variance equation and the mean tracer budget while adjusting θ and θ_2

$$\overline{\phi}_t + \nabla(B_2 - \theta)\cdot\underset{\neg}{\nabla}\bar{b} \;=\; \overline{b'Q'} - K|\nabla\bar{b}|^2 - \nabla\cdot K_2\nabla\bar{b} \tag{22}$$

$$\bar{b}_t + (\bar{\mathbf{u}} - \underset{\neg}{\nabla}B)\cdot\nabla\bar{b} \;=\; \bar{Q} - \nabla\cdot K\nabla\bar{b} \tag{23}$$

The choice $\theta = B_2$ and θ_2 such that $K_2 = 0$ gives the following set of equations, which resemble a generalized TRM (TRM-G hereafter) set of equations

$$\begin{aligned} \bar{\phi}_t &= \overline{b'Q'} - K|\nabla\bar{b}|^2 && (24) \\ \bar{b}_t + (\bar{\mathbf{u}} - \underset{\neg}{\nabla} B)\cdot\nabla\bar{b} &= \bar{Q} - \nabla\cdot K\nabla\bar{b} && (25) \end{aligned}$$

with

$$\begin{aligned} B &= |\nabla\bar{b}|^{-2}(\mathbf{F} - \underset{\neg}{\nabla} B_2)\cdot\underset{\neg}{\nabla}\bar{b} \quad , \quad K = |\nabla\bar{b}|^{-2}(\mathbf{F} - \underset{\neg}{\nabla} B_2)\cdot\nabla\bar{b} && (26) \\ B_2 &= |\nabla\bar{b}|^{-2}(\overline{\mathbf{u}'\phi} - \underset{\neg}{\nabla}\theta_2)\cdot\underset{\neg}{\nabla}\bar{b} \quad , \quad \theta_2 = -\int_{s_0}^{s} [\overline{\mathbf{u}\phi}]_{\perp} ds && (27) \end{aligned}$$

As demonstrated in Appendix B, the difference between TRM-G and the TRM version of Medvedev and Greatbatch (2003) (TRM-M hereafter) is given by assuming that $\theta_2 = 0$ in TRM-M, and to TRM-I (McDougall and McIntosh, 1996) by assuming further that $b_z >> b_y$ and $\overline{\mathbf{u}'\phi} \approx 0$.

The beauty of the TRM-G set of equations is that only one diffusive term with a single diffusivity (given by $K = |\nabla\bar{b}|^{-2}(\bar{\phi}_t - \overline{Q'b'})$) shows up on the right hand side of the eddy variance and mean tracer budget[5]. Furthermore, this particular choice of K is positive (implying downgradient transfer) if there is either growth in eddy variance ($\bar{\phi}_t > 0$) or irreversible removal of eddy variance ($-\overline{Q'b'} > 0$) (for an example of the former situation, but taking b to be potential vorticity rather than potential density, see Wilson and Williams (2004)). The TRM-G decomposition of fluxes was anticipated by Greatbatch (2001), where the normal flux of eddy variance was overlooked. The flux decomposition given by Marshall and Shutts (1981) is also a special case of the TRM-G in which it is assumed that $\theta_2 = K_2 = 0$; that is, that the entire flux of variance is along $\bar{b}$-contours. It should be noted that in the adiabatic, steady limit ($Q = \bar{\phi}_t = 0$) $K = 0$ and the residual flow $\bar{\mathbf{u}} - \underset{\neg}{\nabla} B$ is strictly along contours of $\bar{b}$, showing that the TRM-G is consistent with statement i) in Section 1 (that is, consistent with isopycnal averaging) in the steady, adiabatic limit. What is not clear, however, is how big the eddy-induced diapycnal diffusivity is when $Q \neq 0$, and how well the implied mixing compares with our expectations, based on the presumption that diapycnal transfer and water mass conversion is weak in the ocean interior (an issue we explore using numerical experimentation in Section 3).

Alternatively we can set θ such that $K = 0$ and θ_2 such that $K_2 = 0$ which leads to the

[5] Note that the term $K|\nabla\bar{b}|^2$ can be interpreted as a conversion term between $\bar{\phi}$ and the mean tracer variance $\bar{b}^2/2$, since it shows also up (with opposite sign) in the budget of $\bar{b}^2/2$ - see Medvedev and Greatbatch (2003).

TEM-A set of equations

$$\begin{aligned} \bar{\phi}_t + \nabla(B_2 - \theta) \cdot \underset{\neg}{\nabla}\bar{b} &= \overline{b'Q'} \qquad (28) \\ \bar{b}_t + (\bar{\mathbf{u}} - \underset{\neg}{\nabla}B) \cdot \nabla\bar{b} &= \bar{Q} \qquad (29) \end{aligned}$$

with

$$\begin{aligned} B &= |\nabla\bar{b}|^{-2}(\mathbf{F} - \underset{\neg}{\nabla}\theta) \cdot \underset{\neg}{\nabla}\bar{b} \quad , \quad \theta = -\int_{s_0}^{s} F_\perp ds \qquad (30) \\ B_2 &= |\nabla\bar{b}|^{-2}(\overline{\mathbf{u}\phi} - \underset{\neg}{\nabla}\theta_2) \cdot \underset{\neg}{\nabla}\bar{b} \quad , \quad \theta_2 = -\int_{s_0}^{s} [\overline{\mathbf{u}\phi}]_\perp ds \qquad (31) \end{aligned}$$

Here, the mean tracer and mean tracer variance budgets show no diffusive terms. On the other hand a term shows up in the $\bar{\phi}$ budget, given by the change of the difference between B_2 and θ on a contour of $\bar{b}$, balancing the rate of change and dissipation of variance given by $\overline{Q'b'} - \bar{\phi}_t$. In the steady adiabatic limit, that is $Q = \bar{\phi}_t = 0$, the TEM-A set of equations becomes the same as the TRM-G (Eq. 24 and Eq. 25), since in Eq. 28 we can put $\theta = B_2$ (up to a function of $\bar{b}$). However, for non zero Q there will be a difference between the choices for θ in the TRM-G and the TEM-A, forcing a non-zero K in the TRM-G as demonstrated in Appendix C.

The different eddy tracer flux decompositions

We have presented and named several different versions of the TEM and TRM. These are summarized in Table 1. We have identified the TEM-I version of Andrews and McIntyre (1976) and TEM-II version of Held and Schneider (1999) as special cases of a generalized TEM version, called TEM-G. However, in all these TEM versions, a term related to an apparent eddy–induced diapycnal diffusion shows up in contrast to our aim (statement i) in Section 1) to represent the effect of eddy fluxes as a purely isopycnal flux if $Q = 0$. As shown in Section 3, the implied eddy-induced diapycnal diffusion can become large. Our main result is that it is possible to set this diffusive term to zero by introducing a non–locally defined gauge flux in the eddy tracer fluxes. We call this flux decomposition the adiabatic Transformed Eulerian Mean or TEM-A.

All TEM versions consider the first order moments. In contrast, the TRM versions consider the eddy tracer variance equation to find gauge fluxes for the eddy tracer flux. We have defined a generalized TRM version, called TRM-G, which contains only a single diffusivity K which is given by the (local) rate of change and dissipation of eddy variance (as anticipated by Greatbatch (2001), and for which the decomposition proposed by Marshall and Shutts (1981) is a special case) by introducing a non–local gauge flux in the tracer variance fluxes.

However, in contrast to the TEM-A flux decomposition the diapycnal diffusivity in TRM-G shows up in the mean tracer equation for non zero Q and is related to $\overline{Q'b'}$. The gauge flux of the eddy tracer flux in the formulation is given by the along–isopycnal flux of variance corrected by the gauge flux of the variance fluxes. Setting the gauge flux of the variance fluxes to zero leads to TRM-M, for which the special case $b_z >> b_y$ and $\overline{v'\phi} \approx 0$ leads to TRM-I. We have shown that the TRM-G version approaches TEM-A in the steady and adiabatic limit. This is not true for TRM-M or TRM-I, since here the eddy–induced diapycnal diffusivity does not necessarily vanish in the steady, adiabatic limit. McDougall and McIntosh (1996) show that the diffusivity should be zero to cubic order in perturbation amplitude, an approximation that does not hold up in the numerical experiments which are discussed next.

3 Numerical simulations of eddy fluxes

We proceed now by diagnosing numerical experiments using an OGCM in several idealized configurations with respect to the above presented eddy flux decompositions. The numerical code[6] which is used to integrate the OGCM is based on a revised version of MOM2 (Pacanowski, 1995). We shall explain the specific configurations of the numerical experiments where they appear first in this section.

Experiment CHANNEL0 is a setup with a reentrant channel on a β–plane (referenced to the southernmost latitude of the model domain). Horizontal resolution is $1/3^o$ and there are 20 levels of 100m thickness. Initial conditions are a state of rest and constant meridional and vertical gradients in temperature ($-1 \times 10^{-5} K/m$ and $8.2 \times 10^{-3} K/m$ respectively) and no zonal gradient (except for a small perturbation). A linear equation of state is used ($\partial\rho/\partial T = -0.2 \times 10^{-3} kg/m^3/K$) and salinity set to a constant. Boundary conditions are no slip at the side walls and vanishing heat fluxes at the side walls, surface and bottom boundaries. Bottom friction following a quadratic drag law is used with a coefficient of 1.5×10^3, lateral biharmonic friction with viscosity of $2 \times 10^{11} m^4/s$ and the Quicker advection scheme (Leonard, 1979) for the tracer with no explicit diffusion. Explicit vertical viscosity is $2 \times 10^{-4} m^2/s$. Temperature at the three northernmost and southernmost grid points is relaxed towards the initial condition with a timescale ranging from 3 days at the boundary to 15 days at the outer edge of the relaxation zone.

[6]The numerical code together with all configurations used in this study can be accessed at *http://www.ifm.uni-kiel.de/fb/fb1/tm/data/pers/ceden/spflame/index.html*.

After a couple of weeks integration time, baroclinic instability sets in and is producing large zonal deviations of the flow in the channel. Figure 1 shows the fully developed stage of turbulence after one year integration. The model is integrated for 40 years and temporal and zonal averages are taken for the last 30 years of the integration.

In the following we relate b to temperature, i. e. density, since temperature acts as the only (active) tracer in the model. θ is calculated by interpolating $\mathbf{F}(y, \bar{b}(y,z)) \cdot \nabla\bar{b}/|\nabla\bar{b}|$ to an equidistant grid in the new coordinate $\bar{b}$ and integrating this quantity along lines of constant $\bar{b}$, starting inside the southern relaxation zone, where we put $\theta = 0$. In this way, we obtain $\theta(y, \bar{b}) = -\int_{s_0}^{s} F_\perp ds$ and then interpolate θ back to z as vertical coordinate. This θ serves than as the boundary condition to solve the Euler–Lagrange Eq. 14. (Note that the last step is formally not needed.)

We first compare the eddy streamfunctions and diffusivities in the flux decomposition TEM-G with the decomposition TEM-A. Figure 2 shows eddy streamfunction and eddy–induced diapycnal diffusivity in both cases for experiment CHANNEL-0. Note that in this setup the TEM-I version of Andrews and McIntyre (1976) is virtually the same as TEM-G since $b_y << b_z$ (the TEM-II differs a lot from TEM-G however). The diapycnal diffusivity K in TEM-A fluctuates around zero diapycnal diffusivity, as expected by construction, while K in TEM-G is much larger, about $150\, cm^2/s$ at maximum, resembling a large eddy–induced diapycnal diffusivity.

The values of the eddy streamfunction range from 0 to $-30\, m^2/s$ in both flux decompositions. Note that the streamfunctions are given for the zonally averaged flow and that a value of $30\, m^2/s$ corresponds to about $30\, Sv$ per 10^o longitude volume transport in this model setup. Thus, there is a rather strong eddy–driven meridional overturning in this setup. Since the streamfunction of the mean flow in CHANNEL0 (not shown explicitly) is much smaller compared to the eddy streamfunction, the meridional flow is dominantly eddy–driven in this setup. Note also that in addition to the meridional overturning, there is also a strong zonal transport of about $400\, Sv$ (not shown however).

Figure 3 shows the (residual) streamfunction for the total (meridional) flow $(\bar{\mathbf{u}} - \underset{\neg}{\nabla} B)$ in TRM-G and TEM-A. The residual streamfunction for TEM-G shows strong cross–isopycnal flow while the residual flow of TEM-A is more or less aligned along isopycnals with very little flow across isopycnals, which is entirely driven by $\bar{Q}$. On the other hand, we can conclude that the term $\nabla \cdot K\nabla\bar{b}$ drives the strong cross–isopycnal residual flow in TEM-G and that $\nabla \cdot K\nabla\bar{b}$ is therefore much larger than $\bar{Q}$. Removing this term by accounting for the cross-isopycnal eddy tracer transport $F_\perp$ by adding $\underset{\neg}{\nabla}\theta$ to $\mathbf{F}$ in TEM-A leads to a total flow more or less aligned along isopycnals, removing the large apparently eddy–induced diapycnal diffusivity in TEM-G.

We proceed now by comparing TEM-A and TEM-G with the different TRM versions using output from experiment CHANNEL-0. Figure 4 shows the residual streamfunction and the eddy–induced diapycnal diffusivity K for TRM-M (Medvedev and Greatbatch, 2003). Also shown are the residual streamfunctions for TRM-I (McDougall and McIntosh, 1996) and TRM-II (McDougall and McIntosh, 2001), which can be seen as special cases (for $b_y << b_z$) of TRM-M as outlined in Appendix B. However, note that B (and correspondingly K) in both TRM versions of McDougall and McIntosh are virtually the same as for the TEM-I of Andrews and McIntyre (1976) (this is because there is almost no mean advection of eddy variance in this setup). It is also of interest that there is significant diapycnal flow in both TRM-I and TRM-II versions. Since TRM-II is constructed so as to mimic isopycnal averaging, the implied diapycnal diffusivity in this case indicates the error that arises because formulae in the TRM-II case are given only to quadratic order in perturbation amplitude, higher order terms being neglected.

The eddy–induced diapycnal diffusivity K in TRM-M is slightly smaller than K in TEM-G, however, there is still strong cross–isopycnal transport in the streamfunction of the residual flow. The reason is given by the difference of θ in TRM-M and θ in TEM-A. Although both rotational potentials θ are similar in the interior (not shown), there are strong differences approaching the surface and bottom. As demonstrated in the Appendix C, the change of the difference between both definitions of θ along contours of $\bar{b}$ results in a non–zero K in the TRM version.

The rotational potential θ for TRM-G ($\theta_2 = -\int_{s_0}^{s} \overline{u\phi}_\perp ds$, not shown) is actually rather similar to θ in TRM-M. Consequently, the eddy-induced diapycnal diffusivity K in TRM-G is only slightly reduced compared to K in TRM-M. The difference in K between TRM-G and TRM-M is given by the effect of the cross–isopycnal eddy variance fluxes. It appears that this term is small compared to the term related to dissipation of eddy variance in the present setup. In the steady state, the eddy–induced diffusivity for TRM-G is given by $K = (\nabla\bar{b})^{-2}\overline{Q'b'}$. It follows that in TRM-G the covariance between forcing and density perturbations drives a large diapycnal diffusivity, orders of magnitude larger than the effect of the mean diabatic forcing $\bar{Q}$.

One might argue that the experiment CHANNEL-0 is rather diabatic compared to reality, since the diapycnal mixing is too strong because of low resolution. In fact, since the specified Q outside the relaxation zones is zero, it follows that the diabatic forcing in the model runs, outside the relaxation zones, is due to spurious numerical effects (Griffies et al., 2000; Eden and Oschlies, 2004).

The most effective way to reduce these effects and correspondingly Q in the model is simply given by increasing the resolution. Figure 5 shows B, K and total streamfunction in the different TEM/TRM versions in experiment CHANNEL0-12, in which we have increased the horizontal and vertical model resolution by a factor 4. In addition, we have decreased the biharmonic viscosity to $2 \times 10^{10} m^4/s$ and relax the temperature towards the initial condition in 12 (instead of 3) southern– and northernmost grid points. All other aspects of the model are unchanged with respect to experiment CHANNEL0.

The diapycnal diffusivity K in the TEM-G and TRM-M is reduced with respect to experiment CHANNEL-0 and the total flow is now more aligned along mean isopycnals as before. Again, TRM-M shows lower diffusivities than TEM-G. However, K is of the order of 20 cm^2/s in TRM-M and there are regions of even stronger diapycnal diffusivities, e. g. near the bottom of the channel. These features are further reduced but not eliminated by going to TRM-G (not shown). We might conclude therefore, that the impact of the eddy variance fluxes on the definition of K is stronger in experiment CHANNEL0-12 compared to CHANNEL0 as a result of the reduced dissipation $\overline{Q'b'}$. However, the eddy–induced diapycnal mixing in TRM-G is still orders of magnitude larger than the mean forcing $\bar{Q}$, since we see almost no cross–isopycnal transport in TEM-A in contrast to TRM-G.

4 Discussion

We can ask which of the many eddy tracer flux decompositions summarized in Table 1 is the "best"? In the first section, we formulated statement i) by considering averaging on constant b surfaces (isopycnal averaging when b is potential density, as here). In particular, we argued that a desirable feature of an eddy flux decomposition should be that if the instantaneous diabatic forcing is zero, then there should be no diapycnal flux implied by the mean equations. A possible way to judge the different eddy tracer flux decompositions is to ask how consistent they are with this statement.

TEM-G is certainly the simplest decomposition we have considered, since it is the easiest to understand and to diagnose from data (either model results or observations) and is for strongly stratified situations virtually the same as the TEM of Andrews and McIntyre (1976) (TEM-I). However, the eddy–induced diapycnal diffusivity in this decomposition does not vanish in general for the adiabatic case, which is in clear contradiction to statement i). Only for vanishing cross–isopycnal eddy tracer flux does the diapycnal diffusivity vanish. In the numerical experiments, the diagnosed diapycnal diffusivities for TEM-G are the largest in all the flux decompositions considered.

TRM-I (McDougall and McIntosh, 1996) is similar TRM-M (Medvedev and Greatbatch, 2003), but applied to the case of a strongly stratified ocean (the remaining difference between TRM-I and TRM-M arises because McDougall and McIntosh neglect the triple correlation in the eddy variance fluxes). As for TEM-G, it is not guaranteed that the implied eddy-induced diapycnal diffusivity is zero in the adiabatic limit. (This is because the formulae given by McDougall and McIntosh (1996) neglect terms of cubic order and higher in perturbation amplitude, terms that are not small in our numerical experiments.) Only for the steady case, and if the cross–isopycnal flux of variance vanishes, the eddy–induced diapycnal diffusivity becomes zero when there is no diabatic forcing.

TRM-G and TEM-A both guarantee zero eddy–induced diapycnal diffusivity for the adiabatic and steady case. We might argue therefore that only these two eddy flux interpretations satisfy our statement i) and we may proceed to consider now the more realistic case of weak diabatic forcing. For the steady case of non–zero diabatic forcing, TRM-G gives eddy–induced diapycnal diffusivity related to the covariance between forcing and density. However, the effect of $\nabla \cdot (K \nabla \bar{b})$ from TRM-G is orders of magnitude larger than $\bar{Q}$ in the numerical experiments. We have to assume either very strong eddy–induced diapycnal mixing in the numerical experiments or that the eddy flux interpretation in TRM-G is not in agreement with our implicit belief that eddy–induced diapycnal mixing is a weak effect outside boundary layers. In contrast, the TEM-A flux decomposition as defined in this paper has zero eddy–induced diapycnal mixing by construction, regardless of whether there is diabatic forcing or not, and even if the correlation $\overline{Q'b'}$ is non-zero. It is nevertheless possible to change the minimization problem that leads to TEM-A by allowing for non-zero eddy-induced diapycnal diffusivity, an issue we discuss further in a later paper.

To answer the above question: none of the eddy flux interpretation is "best". However, TEM-A and TRM-G are clearly "better" than the others, with respect to the formal requirement on eddy-induced/enhanced diapycnal mixing expressed by statement i) in Section 1, but we are unable to judge, which one is better. Therefore we may apply another measure to judge the different interpretations: We may criticize that TEM-I, TEM-G, TRM-I/II and TRM-M give *ad hoc* solutions for the interpretation of eddy fluxes. Andrews and McIntyre (1976) simply define an eddy streamfunction without any other justification than that it turns out to be a rather good choice for the stratified situation. The same holds for TEM-G. Medvedev and Greatbatch

(2003) (and accordingly McDougall and McIntosh (1996)) aim to simplify the variance equation by specifying the rotational eddy fluxes in a certain *ad hoc* way, with the justification that this choice might turn out to be in agreement to what they have been in mind. A similar statement holds for TRM-G.

TEM-A, on the other hand, results from a minimal problem, in which we can implement in a objective manner what we believe to be known about the eddy tracer fluxes from the layered framework. Therefore, this decomposition gives a well–defined eddy flux decomposition and interpretation. However, TEM-A is hampered by our current inability to exactly quantify where and by how much eddies enhance diapycnal mixing. This issue will be explored in a later paper.

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Appendix A

In this section, we outline the consequences of temporal instead of zonal averaging on TEM-A. Note that all vectors are three–dimensional in this section. Starting with the instantaneous tracer budget Eq. 1, we decompose b, the velocity $\mathbf{u}$ and the diabatic forcing Q into temporal averages at constant height and deviations

$$b = \bar{b} + b' \quad , \quad \bar{b} = (t_2 - t_1)^{-1} \int_{t_1}^{t_2} b \, dt \quad , \quad b' = b - \bar{b} \tag{32}$$

and analogous for $\mathbf{u}$ and Q. Averaging Eq. 1 yields

$$\bar{b}_t + \nabla \cdot (\bar{\mathbf{u}}\bar{b} + \overline{\mathbf{u}'b'}) \quad = \quad \bar{Q} \tag{33}$$

TEM-A for three dimensions can be written as

$$\mathbf{F} = \begin{pmatrix} \overline{u'b'} \\ \overline{v'b'} \\ \overline{w'b'} \end{pmatrix} = \nabla \times \theta + \mathbf{B} \times \nabla\bar{b} + K\nabla\bar{b} \tag{34}$$

where θ is a three–dimensional vector potential and

$$\mathbf{B} \;=\; -|\nabla\bar{b}|^{-2}(\mathbf{F}-\nabla\times\theta)\times\nabla\bar{b} \quad , \quad K=|\nabla\bar{b}|^{-2}(\mathbf{F}-\nabla\times\theta)\cdot\nabla\bar{b} \tag{35}$$

Note that we have specified $\mathbf{B}\cdot\nabla\bar{b}=0$ analogous to TEM-G in three dimensions. Now we ask again for a vector $\theta=(\theta^{(1)},\theta^{(2)},\theta^{(3)})$ which minimizes

$$\frac{1}{2}\int|\nabla\bar{b}|(K\nabla\bar{b})^2dxdydz=\frac{1}{2}\int L(\theta_z^{(1)},\theta_y^{(1)},\theta_z^{(2)},\theta_x^{(2)},\theta_x^{(3)},\theta_y^{(3)})dxdydz=min \tag{36}$$

The Euler–Lagrange equation is given by

$$(L_{\theta_y^{(1)}})_y+(L_{\theta_z^{(1)}})_z+(L_{\theta_x^{(2)}})_x+(L_{\theta_z^{(2)}})_z+(L_{\theta_x^{(3)}})_x+(L_{\theta_y^{(3)}})_y=0 \tag{37}$$

with

$$L_{\theta_x^{(i)}}=\frac{\partial(\nabla\times\theta\cdot\nabla\bar{b})}{\partial\theta_x^{(i)}}(\mathbf{F}\cdot\nabla\bar{b}|\nabla\bar{b}|^{-1}-\nabla\times\theta\cdot\nabla\bar{b}|\nabla\bar{b}|^{-1}) \tag{38}$$

Now specify for instance $\theta=(\delta,0,0)$ and get

$$(L_{\delta_y})_y+(L_{\delta_z})_z=(-\bar{b}_z(F_\perp-\nabla\times\theta\cdot\nabla\bar{b}|\nabla\bar{b}|^{-1}))_y+(\bar{b}_y(F_\perp-\nabla\times\theta\cdot\nabla\bar{b}|\nabla\bar{b}|^{-1}))_z=0 \tag{39}$$

Thus the term inside the innermost brackets is constant on contours of $\bar{b}$ in the meridional (y–z) plane on which we can integrate the cross–isopycnal flux $F_\perp$ to obtain δ, analogous as before for the zonal mean case. Using the expression for $\overline{\mathbf{u}'b'}$ in the mean tracer budget yields

$$\bar{b}_t+(\bar{\mathbf{u}}+\nabla\times\mathbf{B})\cdot\nabla\bar{b} \;=\; \bar{Q} \tag{40}$$

Note that the same method can be used for TRM-G in three dimensions.

Appendix B

In this appendix we show the relation of TRM-G as presented in this paper with the flux decompositions TRM-I (McDougall and McIntosh, 1996), TRM-II (McDougall and McIntosh, 2001) and TRM-M (Medvedev and Greatbatch, 2003). As discussed in section 2, the choice $\theta=B_2$ and θ_2 such that $K_2=0$ leads to the TRM-G set of equations

$$\begin{aligned}\bar{\phi}_t &= \overline{b'Q'}-K|\nabla\bar{b}|^2 &(41)\\ \bar{b}_t+(\bar{\mathbf{u}}-\underset{\neg}{\nabla}B)\cdot\nabla\bar{b} &= \bar{Q}-\nabla\cdot K\nabla\bar{b} &(42)\end{aligned}$$

Setting instead $\theta = B_2$ and $\theta_2 = 0$, i. e. omitting all non–local gauge fluxes, a term related to the cross-isopycnal flux of eddy variance shows up in the eddy variance equation:

$$
\begin{aligned}
\bar{\phi}_t &= \overline{b'Q'} - K|\nabla\bar{b}|^2 - \nabla\cdot K_2\nabla\bar{b} & (43)\\
\bar{b}_t + (\bar{\mathbf{u}} - \underset{\neg}{\nabla} B)\cdot\nabla\bar{b} &= \bar{Q} - \nabla\cdot K\nabla\bar{b} & (44)
\end{aligned}
$$

with

$$
\begin{aligned}
B = |\nabla\bar{b}|^{-2}(\mathbf{F} - \underset{\neg}{\nabla} B_2)\cdot\underset{\neg}{\nabla}\bar{b} \quad , \quad K &= |\nabla\bar{b}|^{-2}(\mathbf{F} - \underset{\neg}{\nabla} B_2)\cdot\nabla\bar{b} & (45)\\
B_2 = |\nabla\bar{b}|^{-2}\overline{\mathbf{u}\phi}\cdot\underset{\neg}{\nabla}\bar{b} \quad , \quad K_2 &= |\nabla\bar{b}|^{-2}\overline{\mathbf{u}\phi}\cdot\nabla\bar{b} & (46)
\end{aligned}
$$

The latter choice leads to TRM-M and further leads to TRM-I, for which it is additionally assumed that $\bar{b}_z >> \bar{b}_y$. We find for the eddy streamfunction B for $\theta_2 = 0$, $\theta = B_2$ and $\bar{b}_z >> \bar{b}_y$

$$\theta = B_2 \approx -\overline{v\phi}/\bar{b}_z \quad \text{and} \quad B \approx -(\overline{v'b'} + \theta_z)/\bar{b}_z \tag{47}$$

which is the TRM-I streamfunction as given by McDougall and McIntosh (1996) (their Eq. 11) except that they neglected additionally the triple correlation in the eddy variance fluxes; that is, they assumed $\overline{v\phi} \approx \bar{v}\bar{\phi}$. The diffusivity K in TRM-M is given by

$$K = |\nabla\bar{b}|^{-2}(\overline{Q'b'} - \bar{\phi}_t - \nabla\cdot K_2\nabla\bar{b}) \tag{48}$$

and its effect in the mean tracer equation for $\bar{b}_z >> \bar{b}_y$ is given by

$$\nabla\cdot K\nabla\bar{b} \approx \left(\frac{\overline{b'Q'}}{\bar{b}_z}\right)_z - \left(\frac{\bar{\phi}_t}{\bar{b}_z}\right)_z - \left(\frac{1}{\bar{b}_z}\left(\frac{\overline{\mathbf{u}\phi}\cdot\nabla\bar{b}}{\bar{b}_z}\right)_z\right)_z \tag{49}$$

which is the same as $\bar{Q}^{\#} - T_z - \bar{Q}$ in Eq. 12 of McDougall and McIntosh (1996). However, here, we haven't neglected the triple correlation in the flux of variance. The $o(\alpha^3)$ terms in the equations in McDougall and McIntosh (1996) are accounting for the assumption $\bar{b}_z >> \bar{b}_y$ which are obviously not needed in TRM-M, which resembles therefore a generalization of TRM-I to the general situation without assuming $\bar{b}_z >> \bar{b}_y$. Note that in TRM-G the last term related to the cross isopycnal variance flux in the above equation does not show up, since $K_2 = 0$ by construction. Note also that in the steady adiabatic limit, this term does not zero out in general, giving rise for a non–zero K even in this limit.

TRM-II attempts to mimic isopycnal averaging by a truncated (vertical) Taylor expansions of b and $\mathbf{u}$ and leads to a very similar definition of the eddy streamfunction as in McDougall and McIntosh (1996)

$$B \approx -(\overline{v'b'} - \bar{v}_z(\bar{\phi}/\bar{b}_z))/\bar{b}_z \tag{50}$$

and is (consequently) in our numerical experiments very similar to the TRM-I version of McDougall and McIntosh (1996). Note, however that the mean variables $\bar{b}$ and $\bar{\mathbf{u}}$ have been redefined in TRM-II, in such a way as to absorb the cross isopycnal flux of variance (again only the mean flux) into the definition of the new variable, making it difficult to relate the TRM-II with a diffusivity K.

Appendix C

The difference between the rotational potential θ in TEM-A and the θ in TRM-M can be seen by comparing effect in the eddy variance equation by the choices $\theta^* = B_2$ and $\theta_2 = 0$ and $\theta = -\int_{s_0}^{s} F_\perp ds$ and $\theta_2 = 0$ for the steady limit

$$\begin{aligned} 0 &= \overline{Q'b'} - K|\nabla\bar{b}|^2 - \nabla \cdot K_2 \nabla \bar{b} && (51) \\ \nabla(\theta^* - \theta) \cdot \underset{\neg}{\nabla}\bar{b} &= \overline{Q'b'} - \nabla \cdot K_2 \nabla \bar{b} && (52) \end{aligned}$$

Thus by adding both equations we obtain

$$\begin{aligned} \nabla(\theta^* - \theta) \cdot \underset{\neg}{\nabla}\bar{b} &= K|\nabla\bar{b}|^2 && (53) \\ & && (54) \end{aligned}$$

It is the change of the difference between both definitions along contours of $\bar{b}$ which is accounted for by the eddy–induced diffusivity K. The same holds obviously for the differences in θ in TEM-A and TRM-G.

References

Andrews, D. G., J. R. Holton, and C. B. Leovy, 1987: *Middle Atmosphere Dynamics*. Academic Press.

Andrews, D. G. and M. E. McIntyre, 1976: Planetary waves in horizontal and vertical shear: The generalized Eliassen–Palm relation and the zonal mean accelaration. *J. Atmos. Sci.*, **33**, 2031–2048.

Andrews, D. G. and M. E. McIntyre, 1978: Generalized Eliassen–Palm and Charney–Drazin theorems for waves on axisymmetric mean flows in compressible atmosphere. *J. Atmos. Sci.*, **35**, 175–185.

Eden, C. and A. Oschlies, 2004: Effective diffusivities in model of the North Atlantic. Part I: subtropical thermocline. *J. Phys. Oceanogr.*. Submitted.

Gent, P. R., J. Willebrand, T. J. McDougall, and J. C. McWilliams, 1995: Parameterizing eddy-induced tracer transports in ocean circulation models. *J. Phys. Oceanogr.*, **25**, 463–474.

Greatbatch, R. J., 2001: A framework for mesoscale eddy parameterization based on density-weighted averaging at fixed height. *J. Phys. Oceanogr.*, **31**(9), 2797–2806.

Griffies, S. M., R. C. Pacanowski, and B. R. Hallberg, 2000: Spurious diapycnal mixing associated with advection in a z-coordinate ocean model. *Mon. Wea. Rev.*, **128**, 538–564.

Hasselmann, K., 1982: An ocean model for climate variability studies. *Prog. Oceanogr.*, **11**, 69–92.

Held, I. M. and T. Schneider, 1999: The surface branch of the zonally averaged mass transport circulation in the troposphere. *J. Atmos. Sci.*, **56**(11), 1688–1697.

Leonard, B. P., 1979: A stable and accurate convective modelling procedure based on quadratic upstream interpolation. *Computer Methods in Applied Mechanics and Engineering*, **19**, 59–98.

Marshall, J. and G. Shutts, 1981: A note on rotational and divergent eddy fluxes. *J. Phys. Oceanogr.*, **11**(12), 1677–1679.

McDougall, T. J., 1987: Neutral surfaces. *J. Phys. Oceanogr.*, **17**, 1950–1964.

McDougall, T. J. and P. C. McIntosh, 1996: The temporal–residual–mean velocity. Part I: Derivation and the scalar conservation equation. *J. Phys. Oceanogr.*, **26**, 2653–2665.

McDougall, T. J. and P. C. McIntosh, 2001: The temporal-residual-mean velocity. Part II: Isopycnal interpretation and the tracer and momentum equations. *J. Phys. Oceanogr.*, **31**(5), 1222–1246.

Medvedev, A. S. and R. J. Greatbatch, 2003: On transport and mixing in the mesosphere and lower thermosphere: The role of rotational and diffusive fluxes. *J. Geophys. Res.*. In press.

Nakamura, N., 2001: A new look at eddy diffusivity as a mixing diagnostic. *J. Atmos. Sci.*, **58**, 3695–3701.

Nurser, A. J. G. and M.-M. Lee, 2004: Isopycnal averaging at constant height: Part I: the formulation and a case study. *J. Phys. Oceanogr.*. In press.

Olbers, D. and M. Visbeck, 2003: A zonally averaged model of the meridional overturning in the Southern Ocean. *J. Phys. Oceanogr.*. Submitted.

Pacanowski, R. C., 1995: MOM 2 Documentation, User's Guide and Reference Manual. Technical report, GFDL Ocean Group, GFDL, Princeton, USA.

Radko, T. and J. Marshall, 2004: Eddy–induced diapycnal fluxes and their role in the maintenance of the thermocline. *J. Phys. Oceanogr.*, **34**, 372–383.

Tandon, A. and C. Garrett, 1996: On a recent parameterization of mesoscale eddies. *J. Phys. Oceanogr.*, **26**(3), 406–416.

Wilson, C. and R. G. Williams, 2004: Why are eddy fluxes of potential vorticity difficult to parameterize? *J. Phys. Oceanogr.*, **34**(1), 142–155.

Wüst, G., 1935: Schichtung und Zirkulation des Atlantischen Ozeans. Die Stratoshäre des Atlantischen Ozeans. *Wiss. Ergebn. Dt. Atlant. Exped. "Meteor" 1925–1927*, **6**, 109–288.

Acronym	Meaning	Assumption	diapycnal diffusivity
TEM-I	Transformed Eulerian Mean (Andrews and McIntyre, 1976)	$\bar{b}_z >> \bar{b}_y$	$\delta/\bar{b}_z$
TEM-II	Transformed Eulerian Mean (Held and Schneider, 1999)	$\bar{b}_y >> \bar{b}_z$	$\delta/\bar{b}_y$
TEM-G	generalized TEM (Andrews and McIntyre, 1978)	-	$K = \|\nabla\bar{b}\|^{-2}\overline{\mathbf{u}'b'} \cdot \nabla\bar{b}$
TEM-A	Transformed Eulerian but Adiabatic Mean	-	0
TRM-I	Temporal Residual Mean (McDougall and McIntosh, 1996)	$\bar{b}_z >> \bar{b}_y$, $\overline{v'\phi} \approx 0$	$\delta/\bar{b}_z$
TRM-II	Temporal Residual Mean (McDougall and McIntosh, 2001)	$\bar{b}_z >> \bar{b}_y$, $\overline{v'\phi} \approx 0$	see appendix B
TRM-M	Temporal Residual Mean by Medvedev and Greatbatch (2003)	$\theta_2 = 0$	see appendix B
TRM-G	generalized TRM	-	$K = \|\nabla\bar{b}\|^{-2}(\overline{Q'b'} - \bar{\phi}_t)$

Table 1: Summary of names and features of the eddy flux decompositions discussed in the text. Note that the definition for K and δ differ for each version.

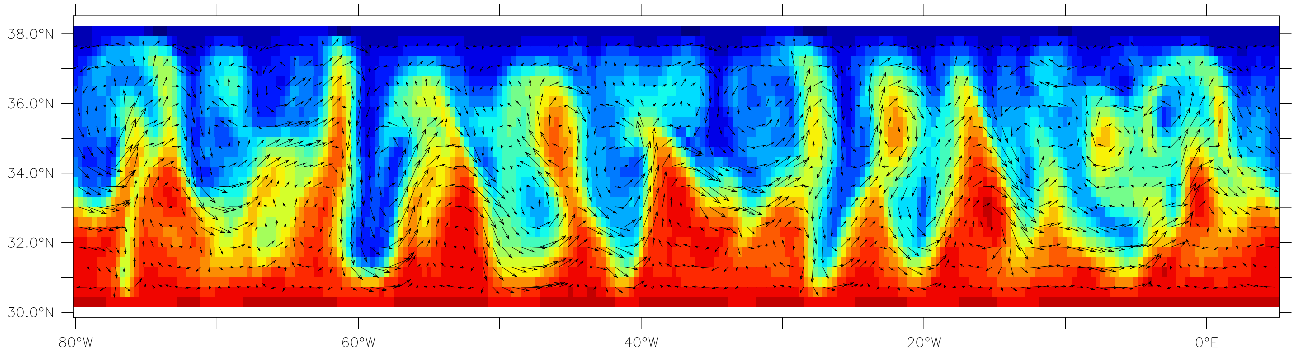

Figure 1: Instantaneous temperature and velocity at 1000m depth after one year integration in experiment CHANNEL0. Every second velocity grid point is displayed and the color shading ranges from 2^oC to 12^oC

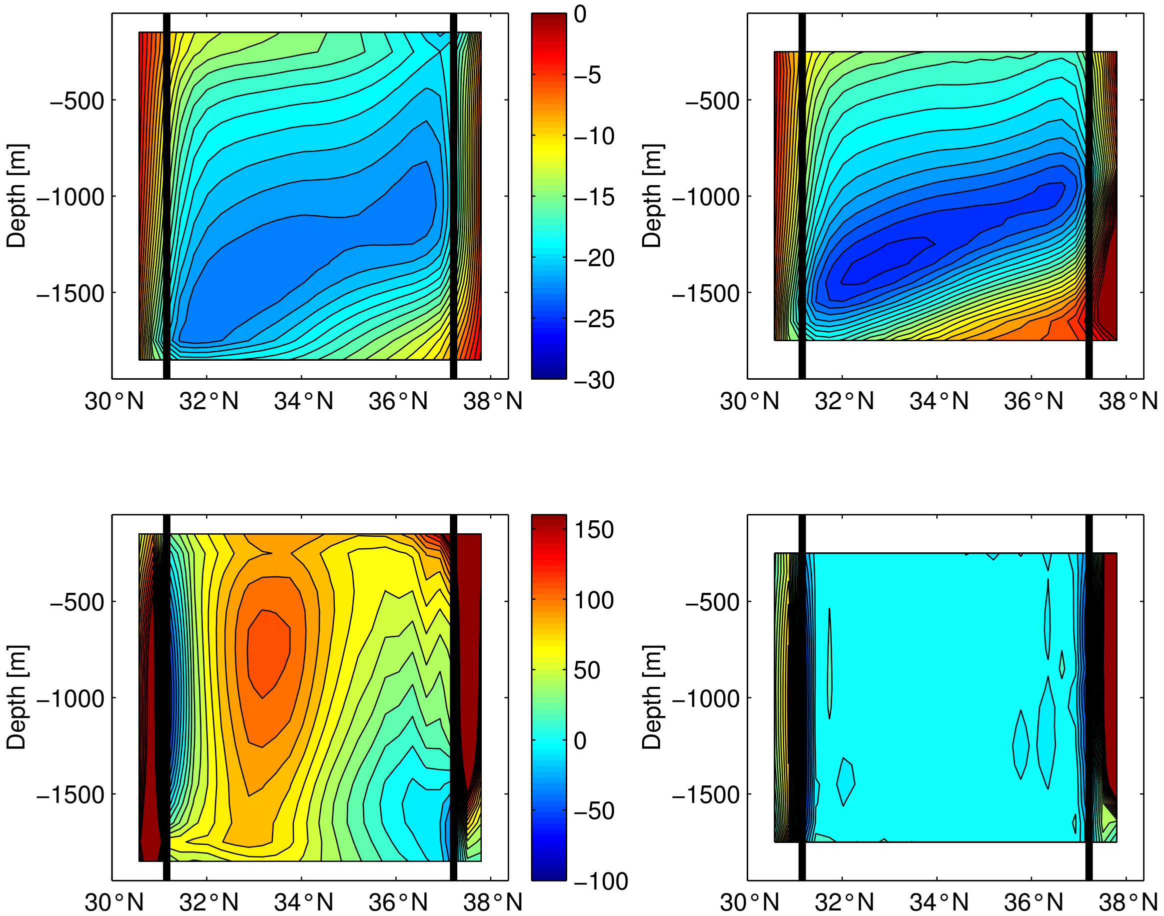

Figure 2: Eddy streamfunction B in m^2/s in TEM-G (upper left panel) and eddy streamfunction B in TEM-A (upper right panel) and diffusivity K in cm^2/s in TEM-G (lower left panel) and K in TEM-A (lower right panel) in experiment CHANNEL0. The vertical bold lines denote the start of the restoring zones in all panels.

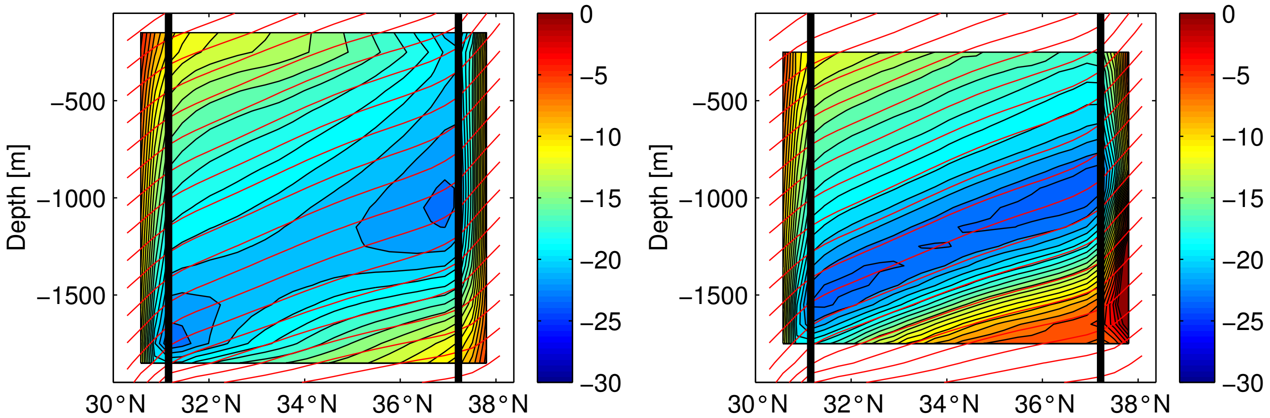

Figure 3: Streamfunction for the total flow in TEM-G (left panel) in m^2/s and in TEM-A (right panel) in experiment CHANNEL0. Also shown are contours of $\bar{b}$ (red lines). The vertical bold lines denote the start of the restoring zones as in all panels.

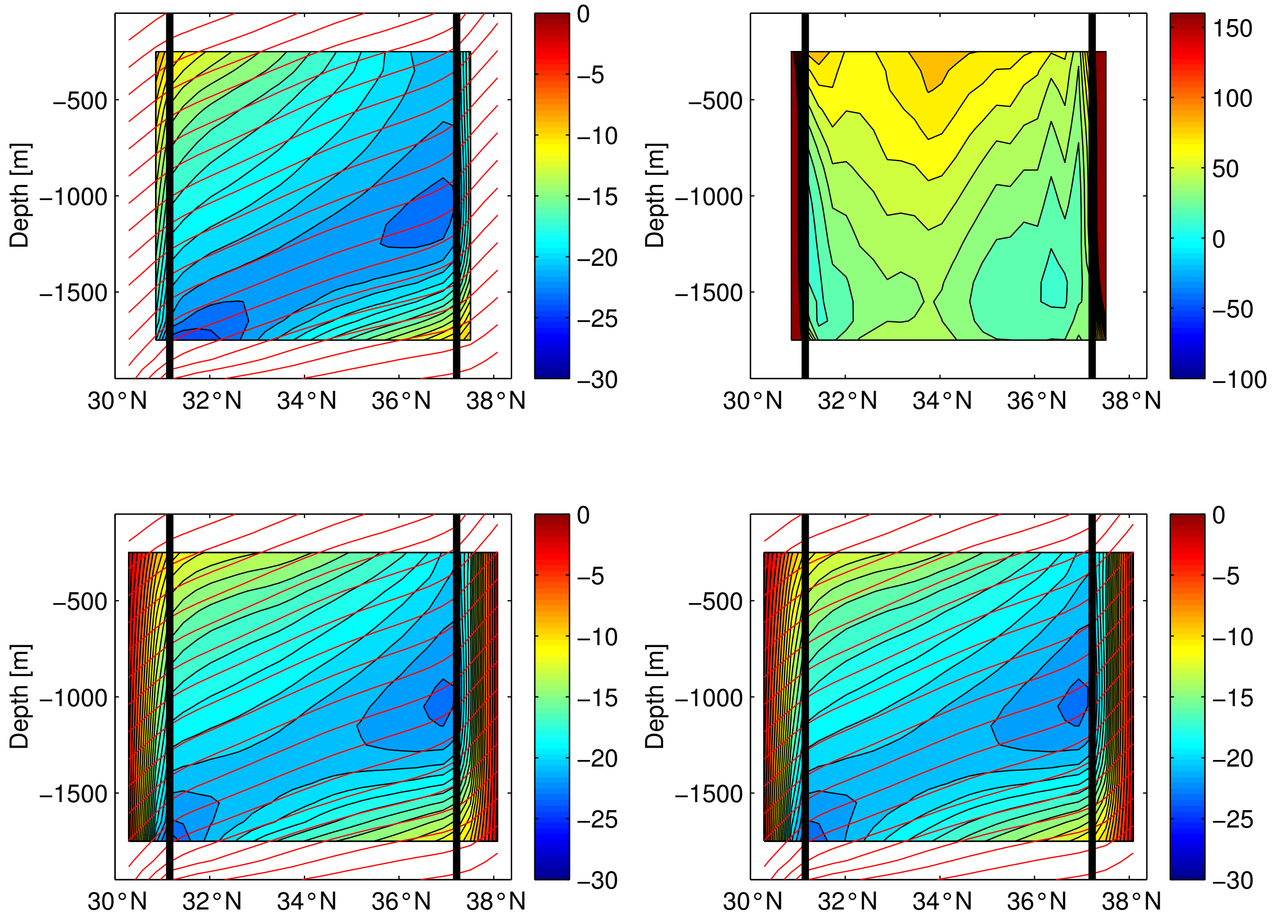

Figure 4: Residual streamfunction B (upper left panel) in m^2/s and diffusivity K in cm^2/s (upper right panel) in TRM-M in CHANNEL0. Also shown are the residual streamfunctions for TRM-I (lower left panel) and TRM-II (lower right panel).

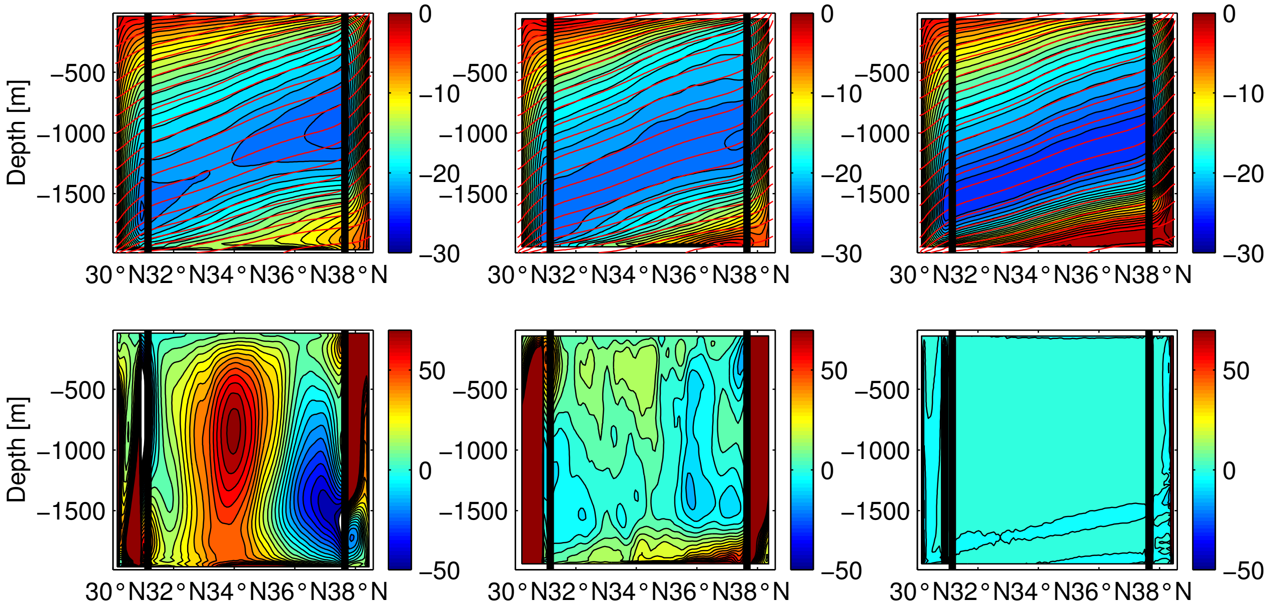

Figure 5: Upper row: residual streamfunctions B in experiment CHANNEL0-12 in m^2/s. Lower row: diffusivities K in cm^2/s. Left column: TEM-G, middle column: TRM-M, right column: TEM-A.