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A note on the viscous attenuation of sound in suspensions

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An expression for the viscous attenuation coefficient in a suspension of solid particles is derived using the viscosity-modified phase shifts for a solid sphere presented in a previous article [Hay and Mercer, J. Acoust. Soc. Am. **78**, 1761–1771 (1985)]. This expression reduces in the appropriate limits to the well-known result for viscous attenuation by small rigid mobile particles.

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In an earlier article,[1] we presented explicit expressions for the phase shifts of the partial waves scattered by a solid elastic sphere, including the effects of the viscosity of the ambient fluid, thus extending Faran's[2] inviscid result. We also gave expressions for these phase shifts for the special case of a rigid mobile scatterer. These expressions are valid for intermediate to short wavelengths, not for the very long-wavelength region where the scattering problem is so much simpler.

It is possible, nevertheless, to show that the viscous absorption coefficient derived using the viscosity-modified phase shifts for the rigid mobile case reduces in the appropriate limits to that obtained by Urick[3] and Lamb.[4] The purpose of this letter is to do so.

When a plane wave is incident upon a spherical target, the scattered wave can be written in terms of a phase shift η_n; that is,[2,5]

$$\hat{p} = -p_0 \sum_{n=0}^{\infty} (2n+1) i \sin \eta_n \, e^{-i\eta_n} h_n(k_c r) P_n(\cos\theta), \tag{1}$$

where η_n is related to the phase shift in the farfield of the nth partial wave of the total field (incident plus scattered), relative to the nth partial wave of the incident wave field. Here, p_0 is the pressure amplitude of the incident plane wave, $\hat{p}$ the complex scattered pressure, r the radial distance from the center of the scatterer, h_n the nth spherical Hankel function of the first kind, k_c the acoustic wavenumber in the fluid, P_n the nth Legendre polynomial, and θ the scattering angle. The harmonic dependence on time, $\exp(-i\omega t)$ with angular frequency ω, has been omitted for convenience.

The viscosity-modified phase shifts take the form[1]

$$\tan \eta_n = \tan \delta_n(x) \left(\frac{\tan \alpha_n(x) + \tan \Psi_n(x',s';s)}{\tan \beta_n(x) + \tan \Psi_n(x',s';s)} \right). \tag{2}$$

Here, x, x', and s' are, respectively, $k_c a$, $k'_c a$, and $k'_s a$; k_c being the compression wavenumber in the fluid; k'_c and k'_s the compression and shear wavenumbers in the solid; and a the radius of the sphere. The argument s is $k_s a$, k_s being the complex wavenumber of the so-called viscous wave,

$$k_s^2 = i\omega\rho_0/\mu_0, \tag{3}$$

where ρ_0 and μ_0 are, respectively, the density and molecular shear viscosity of the fluid. In the inviscid case, the phase shifts have the same form as (2), but $\tan \Psi_n(x',s';s)$ is replaced by $\tan \Phi_n(x',s')$, expressions for which are given elsewhere[1,2] (and in Ref. 6, where they are denoted F_n). The tangent functions in (2) are given by

$$\tan \delta_n(x) = -j_n(x)/n_n(x), \tag{4a}$$

$$\tan \alpha_n(x) = -x j'_n(x)/j_n(x), \tag{4b}$$

$$\tan \beta_n(x) = -x n'_n(x)/n_n(x), \tag{4c}$$

where j_n and n_n are the nth-order spherical Bessel and Neumann functions and the primes denote differentiation.

In the rigid mobile limit ($s',x' \Rightarrow 0$), the $\tan \Psi_n$ reduce to[1]

$$\tan \Psi_n = (n^2+n)/(1+is), \tag{5a}$$

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for $n \neq 1$, and to

$$\tan \Psi_1 = (2\sigma - 3 + is)/(\sigma - 2 + is\sigma), \quad (5b)$$

for $n = 1$, where $\sigma = \rho_0'/\rho_0$, ρ_0' being the density of the solid scatterer. Note that although the above expression for $\tan \Psi_1$ is much simpler than that given in Ref. 1, the two are in fact identical. Proving the identity is straightforward, but involves some algebra. The simplification arises because the denominator of the equivalent expression in Ref. 1 can be shown to be a common factor of the numerator, thus leading to (5b).

The viscous absorption coefficient α_v in a suspension of particles of volume concentration ϵ can be written in terms of the viscosity-modified phase shifts by using[1]

$$\alpha_v = -\frac{3\epsilon}{2k_c^2 a^3} \sum_{n=0}^{\infty} (2n+1)[|A_n|^2 + \mathrm{Re}(A_n)], \quad (6)$$

where A_n is the amplitude of the nth partial scattered wave and is given in terms of the phase shifts by

$$A_n = -i \sin \eta_n \, e^{-i\eta_n}. \quad (7)$$

Now, Urick's result is

$$\alpha_v = (\epsilon k_c/2)(\sigma - 1)^2 \{ s_* / [s_*^2 + (\sigma + \tau)^2] \} \quad (x \ll 1, \beta a), \quad (8)$$

where

$$s_* = 9/4\beta a (1 + 1/\beta a) \quad (9a)$$

and

$$\tau = 1/2(1 + 9/2\beta a); \quad (9b)$$

$\beta a = a(\omega\rho_0/2\mu_0)^{1/2}$, and is therefore equal to $|s|/\sqrt{2}$. Urick's result is valid in the long-wavelength limit, $x \ll 1$, and for $x \ll \beta a$. Our expression for the viscosity-modified phase shifts, on the other hand, is valid for $(|s| = \sqrt{2}\beta a) \gg 1$. We therefore need an approximate form of Urick's result, valid for large βa. Expanding (8) in powers of $1/\beta a$ and dropping terms of order $1/\beta^2 a^2$ or less relative to terms of order unity, we find

$$\alpha_v \simeq \frac{9\epsilon k_c}{2\beta a} \left(\frac{\sigma - 1}{2\sigma + 1} \right)^2 \left[1 + \frac{1}{\beta a} \left(\frac{2\sigma - 8}{2\sigma + 1} \right) \right] \quad (x \ll 1 \ll \beta a). \quad (10)$$

The requirement in (8) and (10) that x be very much less than βa places an upper limit on the frequency. Most fluids satisfy this criterion over a considerable frequency range. In water, for example, the frequency must be much less than 10^{12} Hz. The limit $\beta a \gg 1$ in (10) corresponds physically to requiring that the viscous boundary layer thickness be much less than the scatterer radius, the parameter β being simply the reciprocal of the viscous boundary layer thickness.[1] Since the boundary layer thickness $1/\beta$ is inversely proportional to the square root of the frequency, this condition places a lower limit on the frequency for a given particle size. The expression (10) is therefore valid for what we call here the "high-frequency" end of the long-wavelength region for particles of a given size.

Before deriving an expression for the viscous attenuation coefficient from the phase shifts, it is convenient first to write

$$\tan \eta_n = X_n + iY_n. \quad (11)$$

It can then be shown that

$$\alpha_v = -(9\epsilon/2k_c^2 a^3)\{ Y_n / [(1 - Y_n)^2 + X_n^2] \}. \quad (12)$$

In the long-wavelength limit ($x \ll 1$) we need only be concerned with the $n = 0$ and $n = 1$ terms in the partial wave expansion. Furthermore, the $n = 1$ term is the only one of these two that contributes to viscous absorption [see Eqs. (5)]. Substituting (5b) in (2) and expanding the result in powers of $1/\beta a$ as before, together with $\tan \delta_1 \simeq x^3/3$, $\tan \alpha_1 \simeq -1$, and $\tan \beta_1 \simeq 2$, yields $\tan \eta_1$ in the long-wavelength limit:

$$\tan \eta_1 \simeq -\frac{x^3}{3} \left(\frac{\sigma - 1}{2\sigma + 1} \right) \left[1 - \frac{1}{\beta a} \left(\frac{\sigma - 4}{2\sigma + 1} \right) + \frac{3i}{\beta a} \left(\frac{\sigma - 1}{2\sigma + 1} \right) \right] \left[1 - \frac{1}{\beta a} \left(\frac{4\sigma - 7}{2\sigma + 1} \right) \right]^{-1} \quad (x \ll 1 \ll \beta a). \quad (13)$$

It can be seen that X_1 and Y_1 are both of order x^3 and, therefore, that the denominator in (12) may be set to unity, giving finally

$$\alpha_v \simeq \frac{9\epsilon k_c}{2\beta a} \left(\frac{\sigma - 1}{2\sigma + 1} \right)^2 \left[1 + \frac{1}{\beta a} \left(\frac{4\sigma - 7}{2\sigma + 1} \right) \right] \quad (x \ll 1 \ll \beta a). \quad (14)$$

Comparing this with (10), it is seen that the two results are identical to lowest order in $1/\beta a$. The coefficients of the second-order terms, however, have different numerators. Such differences are to be expected since the higher-order terms will come into play as βa approaches unity, where our result becomes less accurate.

Summarizing, we have shown that the viscosity-modified phase shifts derived in Ref. 1 yield the same absorption coefficient for rigid mobile particles suspended in a fluid as that obtained by Urick, when the latter result is specialized to the high-frequency end of the long-wavelength region: that is, where $k_c a \ll 1$ but the frequency is high enough that the viscous boundary layer thickness is less than the scatterer radius. We conclude that for rigid mobile particles the phase shifts (5) can be used to extend Urick's result beyond the long wavelength limit. For elastic particles, the phase shifts in Ref. 1 must be used.

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