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On the Frontal Speeds of Internal Gravity Surges on Sloping Boundaries

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An expression for the slope dependence of the frontal velocity of an internal gravity surge in deep water is derived from the momentum equations for two-dimensional, inviscid, steady flow. The effects of bottom friction are incorporated by assuming a universal shape for the head of the surge. The resulting expression is consistent with the slope dependence indicated by existing experimental results for small slopes (<5°–15°). The absolute velocity is not predicted by the theory, however, because an indeterminate parameter appears in the result. Furthermore, the experimental data indicate variations in the dependence of nose velocity on slope which depend upon the parameters of the flow itself. It is also shown that the entrainment constant may be estimated from experimental data at large slopes.

INTRODUCTION

The theory of internal gravity surges finds application in many geophysical phenomena, among which turbidity current surges and powder avalanches are important cases. Avalanches are initiated on slopes as steep as 40° and typically come to rest on slopes of 5° or more [see, for example, *Buser and Frutiger,* 1980]. The slopes on which turbidity current surges are initiated are less well known. If, however, one accepts the large body of evidence which indicates that submarine canyons and submarine fan valleys were formed by turbidity currents, and that these events are the cause of submarine cable breaks [*Heezen,* 1963], and the remote acoustic observations of a channelized turbidity surge [*Hay et al.,* 1982], then the typical slopes along the path of a turbidity current surge should range from a few degrees to much less than a degree. The different slopes to be expected for the two phenomena are consistent with the sources and mechanisms responsible for the formation of the unstable deposits from which the events arise. Heavy snowfalls occur at high elevation on steep mountain slopes. Heavy sedimentation occurs on river deltas (and occurred in the geologic past at the edges of continental shelves) of modest or small slopes. The magnitude of the bottom slope represents an interesting dynamical difference between the two phenomena.

There are two classes of internal gravity surge on a sloping bottom: (1) the inclined starting plume (a surge followed by continuous flow), and (2) the inclined thermal (a finite length surge generated by a discontinuous source). Recent work by *Hopfinger and Tochon-Danguy* [1977] and *Beghin et al.,* [1981] on inclined thermals, and by *Britter and Linden* [1980] on inclined starting plumes, has resulted in experimentally verified expressions for the frontal velocities of these flows for bottom slopes from 5° to 90°. The effects of bottom friction were shown to be unimportant for these slopes, the important balance being between the momentum or energy lost in entraining ambient fluid and that gained from the component of the gravitational acceleration parallel to the bed. Data for surges on more gradual slopes did not satisfy the theory because of the increasing importance of viscous effects with decreasing slope. The purpose of this article is to show that an expression consistent with available experimental data for small slopes may be obtained relatively simply by extending the theory of *Benjamin* [1968], including bottom friction, to the case of an inclined bottom in an ambient fluid of infinite depth. It is assumed that no mixing occurs between the two fluids. Although the emphasis is on starting plumes, the theory is also to some extent applicable to inclined thermals. An earlier version of this work was presented in *Hay* [1981].

I have not attempted to present a detailed review of the literature in this introduction. The reader is referred to the recent article by *Simpson* [1982] for a complete and current review.

THEORY

The two-dimensional, inviscid theory of internal gravity surges on a horizontal bottom and followed by a continuous steady flow of uniform density has been presented by *Benjamin* [1968]. The result for the frontal velocity u_0 of a surge with maximum head thickness H_0 is given by

$$u_0 = (g'H_0)^{1/2} \tag{1}$$

provided that the thickness of the continuous flow following the head is equal to $H_0/2$ as indicated in flume experiments [e.g., *Middleton,* 1966]. The parameter g' is $\Delta\rho\, g/\rho_0$, where g is the acceleration due to gravity, $\Delta\rho$ the excess density of the surge, and ρ_0 the density of the ambient fluid (Figure 1*a*). An identical result was obtained earlier by *von Kármán* [1940], but as *Benjamin* [1968] observed, by the incorrect application of Bernoulli's theorem to the interface between the surge and the ambient fluid.

Because the shear stress at the bed is nonzero, the fluid in the surge immediately next to the bed is retarded and an overhang or 'nose' is produced (Figure 1). *Benjamin* [1968, p. 243] incorporated this frictional effect by showing that (1) could be rewritten in terms of a coefficient C_0,

$$C_0 = u_0/(g'H_0)^{1/2} \tag{2}$$

where the numerical value of C_0 depends upon the height of the nose above the bottom. The details of this approach are given below. Furthermore, if the nondimensional shape of the head of a gravity surge were universal, as indicated by

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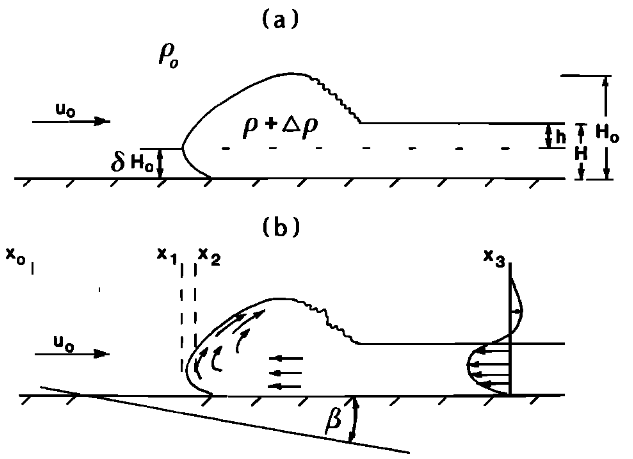

Fig. 1. Schematic diagram of a gravity surge (*a*) on a horizontal bottom and (*b*) on a sloping bottom. Except for the velocity profile at x_3, which represents the following flow, and for which the velocities are relative to the bed, velocities and coordinates are relative to the nose. In the rotated coordinate system (Figure 1*b*), H, H_0, h, and δ are defined as in Figure 1*a*, and the free surface (not shown) is inclined at an angle β to the bottom.

experiment at the time [e.g., *Middleton,* 1966], then C_0 would have a value of 0.77. This is very close to the average value of 0.75 obtained by *Keulegan* [1957, 1958] and *Middleton* [1966] from their experimental results for saline gravity surges. *Middleton* [1966] made experiments at slopes from 0° to 2.3°, and although some dependence of u_0 on bottom slope was noted, this dependence was small and has led to the suggestion that C_0 is essentially constant for turbidity currents [e.g., *Komar,* 1977].

Benjamin [1968] derived equation (1) for a horizontal bed in a frame of reference moving with the nose. In the absence of mixing between the two fluids, the pressure is taken to be constant at the level of the nose ($z = \delta H_0$, Figure 1*a*) within the denser layer. Equation (1) is obtained by invoking Bernoulli's theorem to obtain the pressure in the external flow at the nose stagnation point, and assuming the pressure to be hydrostatic at sufficiently large distances both upstream and downstream of the nose and head region. It is the difference between these hydrostatic pressures which drives the flow and which is balanced by the stress between the fluids associated with the breaking head wave. Friction at the bed is included implicitly since $\delta \neq 0$. Since the pressure must be continuous across the nose, equation (2) is obtained where

$$C_0 = (1 - 2\delta)^{1/2} \tag{3}$$

provided that the thickness H of the continuous flow following the head is half the head thickness H_0 (see Figure 1*a*), as indicated by the flume experiments. These experiments also suggested a universal nondimensional profile of a gravity surge head, with δ approximately equal to 0.2, giving $C_0 =$ 0.77 from equation (3). The universality of head shape is discussed further below.

This theory can be extended to a sloping bottom. Referring to Figure 1*b* and assuming nonlinear, inviscid two-dimensional flow, the downslope force $\rho_0 g \sin\beta$ in the ambient fluid is balanced by the pressure gradient due to the free surface slope in the rotated coordinate system. The pressure difference between x_1 and x_0 is

$$p_1 - p_0 = \rho_0 u_0^2/2 \tag{4}$$

from the integral with respect to x of the conservation equation for the component of momentum parallel to the bed. The points x_3 and x_0 are assumed to be sufficiently far upstream and downstream of the nose that accelerations perpendicular to the bed vanish and the pressure is hydrostatic. The pressure difference between these points is then

$$p_3 - p_0 = \Delta\rho g h \cos\beta \tag{5}$$

where

$$h = H - \delta H_0 \tag{6}$$

assuming that the flow speed differs very little from u_0 at a sufficient height above the head between x_0 and x_3. This limits the argument to cases where the head thickness H_0 is much less than the water depth. Unlike the case of a horizontal bottom, there is now a pressure gradient parallel to the bottom within the denser layer, which in the absence of motion in the denser layer is given by the component of gravity acting parallel to the slope on the excess density,

$$p_2 - p_3 = \Delta\rho g \sin\beta |x_2 - x_3| \tag{7a}$$

Equation (7*a*) has the more general form

$$p_2 - p_3 = \frac{1}{2}\rho'(u - u_0)^2 + \Delta\rho g \sin\beta\, |x_2 - x_3| - \frac{1}{\rho'}\int_{x_2}^{x_3} \frac{\partial}{\partial z}\tau_x\, dx \tag{7b}$$

where ρ' is the density of the lower layer and u and τ_x are the speed of the following flow and the shear stress at the level of the nose stagnation point. *Britter and Linden* [1980] took $u \neq u_0$ and assumed that the second term in (7*b*), which represents the work done by the component of the gravitational acceleration parallel to the bed, was exactly balanced by energy losses due to friction and entrainment, represented by the last term in (7*b*). In the present case it is assumed that the two fluids do not mix and therefore that $u = u_0$. In the present reference frame the following flow is motionless and the third term in (7*b*) vanishes.

Since the pressure must be continuous across the nose, $p_1 = p_2$ and equations (4)–(7) reduce to

$$u_0^2 = 2g'[h\cos\beta + |x_2 - x_3| \sin\beta] \tag{8}$$

If it is assumed that $H_0 = 2H$, then using (3) and (6) this further simplifies to

$$u_0^2 = g'H_0\left[C_0^2 \cos\beta + 2\,\frac{|x_2 - x_3|}{H_0}\sin\beta\right] \tag{9}$$

For small slopes this becomes

$$u_0 = \sqrt{g'H_0}\left[C_0^2 + \frac{|x_2 - x_3|}{H_0}\sin\beta\right] \tag{10}$$

Note that it is C_0 which implicitly incorporates the effects of the bed shear stress.

Comparison With Experiment

Equation (10) conforms reasonably to the slope dependence of $u_0/(g'H)^{1/2}$ obtained in flume experiments, as indicated in Figures 2 and 3. The data in Figure 2 have been taken from *Middleton* [1966, Table II]. Also shown in the figure are the best-fit straight lines obtained by *Shwartz et al.*

[1973] for saline and turbidity thermals. The data in Figure 3 were obtained graphically from Figures 3 and 4 in *Britter and Linden* [1980].

Middleton's [1966] data indicate a decrease in the slope dependence of $(u_0/g'H_0)^{1/2}$ with increasing discharge in the following flow, which suggests that the ratio $|x_2 - x_3|/H_0$ also decreases with increasing discharge. The length $|x_2 - x_3|$ is the distance from the nose to the point in the denser layer at which the pressure becomes hydrostatic. These data suggest that this distance, nondimensionalized with the maximum head thickness, decreases with increasing discharge. In the context of the present theoretical development there seems to be no obvious way to determine this length scale.

The results of *Shwartz et al.* [1973] for saline and turbidity thermals are also in reasonable agreement with the form of the slope dependence given by (10). Furthermore, if $C_0^2 = 0.5$ as indicated by both sets of results for zero slope, and $|x_2 - x_3|$ is simply taken as the length of the thermal (about $2H_0$ in these experiments), then (9) reduces to

$$u_0 = (g'H_0/2)^{1/2}[1 + 4 \sin \beta] \tag{11}$$

This is to be compared with the empirical result of *Shwartz et al.* [1973],

$$u_0 = (g'H_0/2)^{1/2}[1 + 1.6 \sin \beta] \tag{12}$$

indicating that $|x_2 - x_3|$ is actually shorter than the length of the thermal. Experiments with slugs of different volume, which might have shown a change in the slope dependence of u_0 similar to that with discharge in Middleton's work, were not conducted. *Shwartz et al.* [1978] also present a theoretical result similar to (10). No details of the theory were given.

Britter and Linden [1980] found that the nondimensional frontal velocities of inclined starting plumes increased to a maximum at slopes of 20°–30° and then decreased continuously on higher slopes. The values for u_0 at small slopes (0°–10°) taken from their Figures 3 and 4 are plotted against sin β

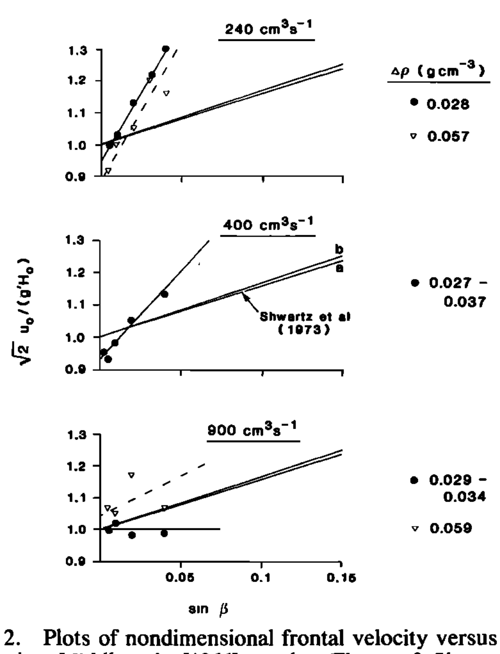

Fig. 2. Plots of nondimensional frontal velocity versus bottom slope using *Middleton*'s [1966] results. Those of *Shwartz et al.* [1973] are shown by solid lines; (a) saline thermals; (b) turbidity thermals.

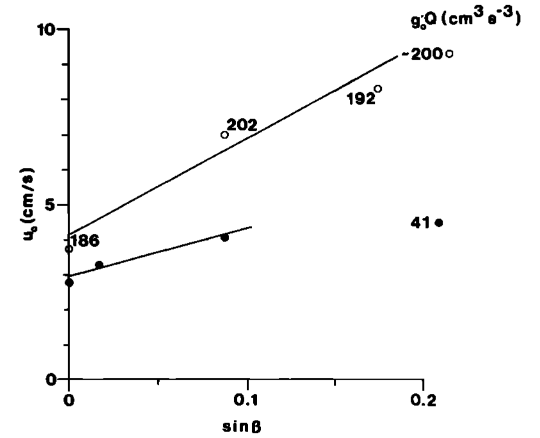

Fig. 3. Frontal speed versus sin β from the experimental results of *Britter and Linden* [1980]. The solid circles are those points for a buoyancy flux/unit width of 41 cm^3 s^{-3}; the open circles represent a flux/unit width of about 200 cm^3 s^{-3}. The actual values of g'_0Q for each point are indicated on the graph.

in Figure 3 of this paper. These data are also consistent with the form of equation (10), although there are not a large number of points. A dependence of u_0 on discharge and/or buoyancy flux is also indicated. Values of $\Delta\rho$ and H_0 were not given.

DISCUSSION

Because it is assumed that no mixing of the two fluids occurs, and that shear stress at the interface is negligible, the following flow moves with the same speed as the nose. In a frame of reference moving at the frontal velocity, the following flow is therefore stationary and justifies keeping only the sin β term in the expression for the down-slope pressure difference at the level of the nose (equations (7*a*) and (7*b*)). Friction at the bed causes the overhanging nose, and the effect of the bed shear stress is incorporated into the inviscid theory as the ratio δ of nose height to head thickness H of the following flow as a constant fraction, taken here to be 0.5, of the head thickness. The same assumptions were made by *Benjamin* [1968] for surges on a horizontal bottom.

Implicit in the assumptions above concerning the geometry of the head is that the nondimensional profile of the head be universal in shape over the range of slopes considered. *Middleton*'s [1966] experiments indicated that the nondimensional profile remained unchanged for slopes ranging from 0° to 3.4°. *Simpson* [1972], on the other hand, obtained the empirical relationship

$$\delta = 0.61 \, Re^{-0.23} \tag{13}$$

where Re is the Reynold's number for surges on a horizontal bed. The dependence of δ on frontal speed is quite weak. If equation (13) should also prove to hold for small slopes, then it could be used iteratively with equations (3) and (9) if necessary.

It has been shown by *Britter and Linden* [1980] and *Beghin et al.* [1981] that a universal nondimensional shape does not exist for either inclined starting plumes or thermals over the entire range of slopes. The ratio of head height to head length decreases at large slopes as the slope increases. In the present context this means that the ratio $|x_2 - x_3|/H_0$ in equation (10) must be slope dependent for large slopes.

Equations (9) and (10) are in any case not useable for steep slopes not only because of the geometrical assumptions made in their derivation, but also because entrainment has not been included.

It is useful to compare the result obtained by *Britter and Linden* [1980] for starting plumes on steep slopes to that presented here for more gradual slopes. Their result is

$$u_0 = (g_0'Q)^{1/3}S_2^{1/3}\left[\frac{\cos\beta}{\alpha} + \frac{\alpha\sin\beta}{2(E + C_D)}\right]\left(\frac{E + C_D}{\sin\beta}\right)^{2/3} \quad (14)$$

where $g_0'Q$ is the buoyancy flux per unit width, S_2 is a shape factor defined by *Ellison and Turner* [1959], α is the ratio of the speed at the level of the nose in the following flow to the mean speed of the following flow, C_D is a drag coefficient, and E is the entrainment rate in the following flow. Of these parameters, E is the most sensitive to bottom slope. *Bo Pedersen* [1980, p. 102] has suggested that

$$E = K\sin\beta \quad (15)$$

For small slopes $E \ll C_D$, and (14) reduces to

$$u_0 = (g'Q)^{1/3}S_2^{1/3}\left[\frac{\cos\beta}{\alpha} + \frac{\alpha\sin\beta}{2C_D}\right]\left(\frac{C_D}{\sin\beta}\right)^{2/3} \quad (16)$$

This equation has a much stronger dependence on slope than indicated by the data and cannot be reduced to a form like equation (10).

For steep slopes, $E \gg C_D$ and substitution of (15) in (14) yields

$$u_0 = (g'Q)^{1/3}S_2^{1/3}\left[\frac{\cos\beta}{\alpha} + \frac{\alpha}{2K}\right]K^{2/3} \quad (17)$$

which shows the decreasing speed with increasing slope indicated by the experiments for $\beta > 30°$, provided α and S_2 remain essentially constant.

Equation (17) can be used as a rough test of the value of K. From *Britter and Linden* [1980, Figure 7], at $\beta = 90°$

$$\frac{1}{22} < \frac{K}{\alpha^3 S_2} < \frac{1}{14}$$

with $\alpha = 1.2$ [*Britter and Linden,* 1980] and $S_2 = 0.7$ [*Ellison and Turner*, 1959], $0.055 < K < 0.086$ which is close to the value of 0.072 proposed by Bo Pedersen.

SUMMARY AND CONCLUSIONS

By extending *Benjamin*'s [1968] theory for inviscid, two-dimensional gravity surge flow to the case of an inclined bottom, and incorporating the effects of bottom friction in the same fashion as was done by Benjamin, it has been possible to reproduce the form of the slope dependence of the frontal velocities of starting plumes shown by experiments on inclines of less than 5° to 15°.

The theory is limited, however, by an indeterminate parameter $|x_2 - x_3|/H_0$ in the final result (equation (9)). Furthermore, the experimental results indicate that the value of this parameter depends upon the properties of the flow itself, such as the magnitude of the volume transport in the following flow.

An approximate form for the frontal velocity on steep inclines is obtained by simplifying *Britter and Linden*'s [1980] result by means of *Bo Pedersen*'s [1980] relation for the entrainment rate, with which the data of *Britter and Linden* [1980] appear to be consistent.

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