Oceanic Thermohaline Intrusions: Theory

Corresponding author: Barry Ruddick,

Department of Oceanography,

Dalhousie University,

Halifax, N.S., B3H 4J1,

Canada.

email: barry.ruddick@dal.ca

Fax: (902) 494-2885

Oliver Kerr,

Department of Mathematics,

City University,

Northampton Square, London EC1V 0HB,

United Kingdom

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Abstract

This is a review of theories governing growth and evolution of thermohaline intrusive motions. We discuss theories based on eddy coefficients and salt finger flux ratios, and on molecular Fickian diffusion, drawing relationships and parallels where possible. We discuss linear theories of various physical configurations, effects of rotation and shear, and nonlinear theories.

A key requirement for such theories becoming *quantitatively* correct is the development and field testing of relationships between double-diffusive fluxes and average vertical gradients of temperature and salinity. While we have some ideas about the functional dependencies and rough observational constraints on the magnitudes of such flux/gradient relationships, many questions will not be answered until usable "flux laws" exist. Furthermore, numerical experiments on double-diffusive intrusions are currently feasible, but will have more quantitative meaning when fluxes are parameterised with such laws.

We conclude that more work needs to be done in at least two areas. First, tests of linear theory against observations should continue, particularly to discover the extent to which linear theories actually explain the genesis of intrusions. Second, theoretical studies are needed of the nonlinear effects that control the evolution and finite amplitude state of intrusions, since these determine the lateral fluxes of salt, heat, and momentum.

1. Introduction

Inversions in temperature and salinity occur in most oceanic CTD casts, and are a signature of thermohaline intrusions, produced by lateral sheared advection across lateral water mass gradients. They are typically "thermohaline" in origin – self-driven by the release of potential energy via vertical double-diffusion, and cause lateral mixing that is slow and steady but comparable to stirring by baroclinic eddies (cf. Joyce et al, 1978).

The dynamical mechanism behind thermohaline intrusions is simple but subtle, and was first elucidated by Stern (1967) and later in laboratory experiments by Turner (1978). Consider a situation with lateral gradients of temperature and salinity (figure 1), and a vertical stratification that supports salt fingering. If a perturbation consisting of alternating shear zones is superposed, the lateral advection and lateral T/S gradients act to produce alternating vertical T and S gradients that will alternately enhance and weaken the existing salt fingers. This produces flux convergences that tend to reduce the T and S perturbations. However, because the buoyancy flux for salt fingers is upgradient (a downward density flux), the fluxes will make the warm, salty perturbations become less dense and the cool, fresh perturbations more dense. If the initial perturbation has a slight slope (as shown) such that the warm, saline perturbations slope upwards from the warm, salty side, then the density perturbations will act to reinforce the initial motion. The warm salty layers thus become anomalously light due to the flux convergence, and "slide upwards" from the warm salty side, with the converse occurring to the cool fresh layers. The linear instability works via a positive feedback loop:

1. lateral, along-intrusion, sheared advection

2. alternately strengthened and weakened gradients and salt finger fluxes

3. alternately positive and negative density perturbations

4. sloping density perturbations create pressure perturbations

5. pressure perturbations accelerate the original advective motions.

Since velocity is anti-correlated with S and T perturbations, the lateral intrusive heat and salt fluxes are downgradient. Since there is a systematic density perturbation, there is an along-intrusion (and slightly downwards) density flux towards the warm, salty side. If the diffusive fluxes dominate, the slope and along-intrusion density flux are reversed.

Intrusions cause significant lateral fluxes, and a main goal is to provide a parameterisation for these fluxes. The aim of most intrusion theories is to understand the mechanisms of formation, growth, evolution, and eventual finite-amplitude limitation, since these affect the lateral fluxes created.

The main factor preventing a *quantitative* understanding is the fact that we cannot yet successfully predict or parameterise the vertical fluxes in a double-diffusive oceanic environment. In an environment with sharp interfaces and well-formed layers, laboratory laws for double-diffusive and turbulent entrainment fluxes are well-established (cf. Turner, 1973), although observational tests in the C-SALT experiment have been problematic (Schmitt, 1994). However, layer formation and growth in a turbulent, sheared oceanic environment, or double-diffusive fluxes in a more uniform gradient, are not well understood, and it is fair to say that no *fully tested* parameterisations exist to predict average vertical fluxes in terms of average gradients. Despite this, a great deal of progress has been made using eddy diffusivity parameterisations building on that of Stern (1967) as well as molecular diffusivities. In the oceans the use of molecular diffusivities to analyse the development of intrusions that are of order 10m thick is certainly inappropriate. So why should oceanographers be interested in this body of work? One reason is that the results from the two approaches often have strong parallels, with the results in one field having direct analogues in the other. Another reason is that the sequence of instabilities leading to intrusions begins with those involving molecular fluxes. In the following sections we summarize many of the laboratory experiments that motivated the studies, look at linear theory based on eddy diffusivites, compare this with theory using molecular diffusivities, look at the effects of rotation and baroclinic shear and then discuss finite amplitude theories.

2. Linear theories

2.1 Linear theories based on eddy diffusivity parameterisation

Uniform gradient configurations

Linearized instability theories predict exponential growth, and therefore give no inherent limitations on the eventual amplitude or lateral fluxes. However, intrusion scales, slopes, physics and growth rates can be predicted, provided the vertical fluxes, double-diffusive and otherwise, can be accurately parameterised. Stern's (1967) vertical diffusivity parameterisation incisively captured the major effect of salt fingers. He parameterised the vertical salt flux βF_S by an eddy diffusivity, K_S, and linked the heat flux αF_T to the salt flux (rather than the temperature gradient) via a flux ratio, γ_f , which is, for salt finger mixing, less than 1:

$$\begin{aligned} \beta F_S &= K_S \beta \frac{\partial S}{\partial z} \\ \alpha F_T &= \gamma_f \beta F_S \end{aligned} \qquad (1.)$$

where α and β are the thermal expansion and haline contraction coefficients, respectively.

Stern (1967) considered the linearized growth of laterally-sloping perturbations in infinite vertical and lateral T and S gradients with horizontal isopycnals. Stern's instability theory showed how lateral intrusions could be driven by salt finger flux convergences. The theory was groundbreaking because, in addition to discovering a new mechanism to drive lateral fluxes, it showed how the intrusions must slope relative to isopycnals, and estimated the growth rate. Stern also clearly showed that turbulent mixing that produces equal turbulent diffusivities for salt and heat, cannot drive intrusions.

Stern's (1967) parameterisation ignored viscosity, and found the growth rate to increase without bound with the vertical wavenumber – a "blue catastrophe". Toole and Georgi (1981) added a constant eddy viscosity, *A*, to Stern's theory, and found the growth rate to maximize at a particular wavenumber, leading to a prediction for the vertical wavelength (i.e., twice the thickness) of the fastest-growing intrusions:

$$H = 2\pi\left[\frac{4(K_S A)^{1/2} N}{g(1-\gamma_f)\beta\bar{S}_x}\right]^{1/2}, \tag{2.}$$

where N is the buoyancy frequency, γ_f the salt finger flux ratio, and $\beta\bar{S}_x$ is the lateral salinity gradient, converted to density equivalent. The growth rate scales roughly as the diffusion time, H^2/K_S. The scales and growth rates depend on the unknown eddy diffusivities for salt and momentum, which are not well-constrained, but reasonable values tend to predict scales in fair agreement with observations.

Joyce et al (1978), Posmentier and Hibbard (1982), and Posmentier and Kirwan (1985) noted that double-diffusive interleaving can cause large lateral fluxes of salt, heat, and mass, and that the density flux must have an upgradient component. McDougall (1985a) considered these effects quantitatively and how the vertical component of the density flux compared with the vertical double-diffusive fluxes between layers.

Walsh and Ruddick (1995a) noted that the perturbations to the salinity and temperature gradients also cause variations in the density ratio (see fig. 1) that are significant even in the linearized theory. Since salt finger fluxes increase with decreasing density ratio, the effect of these perturbations is to significantly enhance flux convergences. Walsh and Ruddick (1995a) show how this effect is equivalent to an enhanced salt diffusivity, with an enhancement of a factor of about 20 for density ratio of about 1.6. For parameters appropriate to Meddy Sharon (Armi et al, 1989), this resulted in a doubling of the fastest-growing layer thickness, and a quadrupling of the growth rate. They also show that eddy viscosity dependence on density ratio does not enter linearized theories to first order. Walsh and Ruddick (1995b, 2000) considered the effects of similar dependence of the finger flux ratio on density ratio, and demonstrated an instability similar to that of Huppert (1971) that may aid formation of layers. Walsh and Ruddick (2000) found that adding a background turbulent diffusivity, with equal turbulent heat and salt diffusivities, supressed this instability by causing the effective flux ratio to increase with density ratio.

Non salt-finger parameterisations

Walsh and Ruddick (1995a) realized that their eddy diffusivity formulation with density ratio dependence could be mapped to the diffusively stratified case (see their section 7b). They compared the predicted intrusion wavelength and "slope", measured as density change per unit salinity change along the layers, to the observations of Ruddick (1993) in both finger and diffusively stratified zones, finding excellent agreement. . As discussed in section 2.2, this formulation can also be used to directly compare with theories based on molecular diffusivities or the case of stable stratification with unequal heat and salt turbulent diffusivities, which can lead to intrusion growth as discussed in the review by Gargett (2001, this issue).

Frontal configurations

The theories of Stern (1967) and Toole and Georgi (1981) had constant vertical and horizontal property gradients in an infinite domain, yielding perturbation equations with constant coefficients. Consequently the medium has no inherent scale other than that defined by diffusivities and stratification parameters (eq. 2). Niino (1986) considered the effects of a finite-width front, with salinity that increased linearly over a finite width D, and was constant outside that region. He found the intrusion properties depended on the magnitude of a dimensionless number loosely analogous to a Rayleigh number:

$$R = \frac{\left(g(1-\gamma_f)\beta\Delta S\right)^6}{K_S A D^2 N^{10}} \tag{3.}$$

He found that a finite front was always unstable to intrusions and identified two regimes: the large-*R* regime and the small-*R* regime. These regimes could be identified in terms of his stratification parameter $\mu = g\left(1-\gamma_f\right)\beta\overline{S}_z / N^2$, with the large-*R* regime prevailing when $R > R_{\min} \approx 2\times10^5(1+\mu)^{4.9}$ and the small-*R* regime when $R < R_{\max} \approx 40(1+\mu)^{5.4}$. When *R* is less than $R_{\max}$ then the front is narrow and the vertical scale of the intrusions is similar to that found by Ruddick and Turner (1979 - see section 3). When R is greater than $R_{\min}$ the front is wide and the vertical scale and growth rates of the intrusions are essentially those predicted by Toole and Georgi with their model which assumed linear gradients of infinite extent.

2.2 LinearTheories based on molecular diffusivities

There are two approaches to parameterising the intrusion-scale fluxes in double-diffusive systems. One is the Stern (1967) eddy-diffusivity parameterisation (section 2.1). The second approach, exemplified by Thorpe, Hutt and Soulsby (1969), uses (Fickian) molecular diffusivities. This approach has several advantages:

1. Molecular diffusivites have known values and behaviour, in contrast to the poorly known gradient flux laws for salt fingers.

2. Such instability theories often can be closely reproduced in laboratory experiments, allowing close observational comparison.

3. In situations with boundaries, such as laboratory tank experiments with heated sidewalls, the appropriate boundary conditions are known.

A disadvantage of molecular diffusivity-based theories is their focus on laboratory scales and phenomena, rather than oceanic, resulting in much smaller predicted layer scales and growth rates. Molecular diffusivities have horizontal as well as vertical fluxes, compared to the vertical fluxes in the Stern parameterisation. Consequently, molecular-based theories such as that of Holyer (1983) allow salt fingers as one mode of instability. It can sometimes become difficult to clearly separate intrusive, fingering, and overstable double-diffusive modes of instability. However, we will see that the results obtained using molecular diffusivities are often closely related to those obtained using different flux laws. Thus an understanding of processes gleaned from studying intrusions ruled by molecular diffusivities may help in understanding processes in the ocean which have different flux laws, and vice versa.

Uniform gradient configurations

Holyer (1983) studied the instabilities of the Toole-Georgi configuration (infinite, density-compensating horizontal T and S gradients, uniform vertical gradients) assuming that the fluxes are driven by molecular diffusivities. She found that the fluid was always unstable to intrusions whenever horizontal temperature and salinity gradients were present. The principal difference is that in this case warm salty water moving towards cooler fresher water will lose heat faster than salt as the diffusivity of heat, κ_T, is much larger than the diffusivity of salt, κ_S. These differences were borne out by observations of opposite sense intrusion slopes in the laboratory experiments of Ruddick and Turner (1979) and of Holyer, Jones, Priestly and Williams (1987). In the Ruddick and Turner experiments the fluxes were dominated by salt fingers, causing the warm salty intrusions to become less dense (see fluid parcel B in figure 1). In Holyer et al (1987) the fluxes were dominated by molecular diffusivities. The dominance of the molecular heat flux over the molecular flux of salt caused warm saline intrusions to lose heat and become more dense, thus causing reversed slope.

If the effects of horizontal diffusion are neglected, it is possible to quantitatively relate the Toole-Georgi (1981) theory to one based on molecular diffusivites, using the Walsh and Ruddick (1995a) model. Walsh and Ruddick (1995b) parameterise fluxes allowing K_S and γ_f to depend on R_ρ:

$$\begin{aligned}\beta F_S &= K_S(R_\rho)\beta\frac{\partial S}{\partial z}\\ \alpha F_T &= \gamma_f(R_\rho)\beta F_S\\ &= \gamma_f(R_\rho)K_S(R_\rho)\beta\frac{\partial S}{\partial z}\\ &= (R_\rho)^{-1}\gamma_f(R_\rho)K_S(R_\rho)\alpha\frac{\partial T}{\partial z}\end{aligned} \tag{4.}$$

In fact, eqs. (4) can be used to map the theory of Walsh and Ruddick (1995b) directly on to that of Holyer (1983), for the case of small intrusion slope. Consider how this parameterisation appears if the diffusivities are in fact molecular, with constant values κ_T and κ_S. If $K_S(R_\rho) = \kappa_S$, which is constant, then the flux ratio must be

$$\gamma_f(R_\rho) = R_\rho \kappa_T / \kappa_S, \tag{5.}$$

which is generally much larger than one and increases with density ratio. The fact that the flux ratio increases with density ratio allows intrusive layers to form in the manner of Toole and Georgi (1981), but results in stable behaviour to the "flux convergence" layer formation mechanism of Walsh and Ruddick (1995b). The fact that the effective flux ratio is greater than one means that the temperature flux exceeds the salinity flux in density terms, causing warm, salty intrusions to become cooler and more dense, resulting in slopes opposite to that in figure 1, and in qualitative agreement with Holyer (1983).

Heated sidewall configurations.

The experiments of Thorpe et al (1969) had well-understood sidewall boundary conditions: fixed wall temperature, a no-slip condition and no salt flux through the vertical walls. In their theoretical analysis they assumed a strong salinity gradient which had the result that the convection consisted of thin almost horizontal layers, and so all fluxes were at leading order vertical. Thus, both Thorpe et al (1969) and Kerr (1989) investigated the stability of stably stratified fluids with lateral temperature and salinity gradients and found thin almost horizontal layered convection with diffusion of heat and salt in the vertical direction only. Thorpe et al. expressed their results in terms of a pair of Rayleigh numbers; however Kerr (1989) showed that the natural parameter for this problem, and for related problems such as heating a salinity gradient from a single side wall, is

$$\begin{aligned} Q &= \frac{(1-\tau)^6 g(\alpha\Delta T)^6}{\nu\kappa_S D^2\left(-\beta\bar{S}_z\right)^5} \\ &= \frac{\left((1-\tau)g(\alpha\Delta T)\right)^6}{\nu\kappa_S D^2 N^{10}} \end{aligned} \tag{6.}$$

where $\tau = \kappa_S / \kappa_T$ is the ratio between the salt diffusivity and the heat diffusivity, ν the kinematic viscosity, D the width of the slot and $\bar{S}_z$ the mean vertical salinity gradient (here $N^2 = -g\beta\bar{S}_z$). Since both Thorpe et al (1969) and Kerr (1989) assumed $\alpha\Delta T = \beta\Delta S$ it can be seen that the stability parameters Q (eq. 6) and Niino's R (eq. 3) are essentially identical.

The physical interpretation of Niino's parameter R may not at first be clear and was not given by Niino, but some insight can be derived by derivation of the form of Q by a mechanistic argument given by Kerr. One difference between the two theories is that for a heated slot in a laboratory there is a critical value $Q = 432\pi^4$ required for instabilities to develop while in the oceans fronts are always unstable. This is related to the fact that the Stern (1967) parameterisation supposes that the T-flux depends on the gradient of S, and not T, so that the T-equation is not "damped" by a diffusive term. In the case of molecular fluxes, both viscosity and diffusion of the density perturbation moderate the process, causing a criticality (Kerr, 1989).

The theory of Kerr (1989) was primarily concerned with intrusions growing at a single sidewall. He assumed that the vertical salinity gradient was strong in the sense that the scale H is much smaller than the horizontal scales of the temperature and salinity gradients. Provided this assumption was valid the theory could be applied to the instantaneous wall temperatures and profiles, ignoring the slow time evolution due to the horizontal diffusion of heat. In this limit the fluxes of heat and salt are, to leading order, purely vertical except in thin boundary layers near the walls. Depending on the rate of heating of the walls it is possible for intrusions to grow even though the final temperature difference corresponds to stability.

A situation where there are not direct analogues between the eddy viscosity and molecular-based parameterisations is in the lateral heating of a salinity gradient in a vertical slot when the vertical gradients are not strong. We have seen that for strong salinity gradients there is a parallel between the instabilities first examined by Thorpe *et al.* and ocean instabilities at a finite front examined by Niino (1986). However, for weaker salinity gradients there are four other modes of instability in a vertical slot (Kerr and Tang, 1999) none of which are likely to have oceanic parallels. They all require horizontal gradients at least as strong as the vertical gradients and give rise to instabilities that have height comparable to or greater than their width.

Melting iceblock configurations.

Huppert and Turner (1980) showed in a series of laboratory experiments that melting ice in salt-stratified water produces a regular sequence of intrusive layers, and found that their thickness is proportional to the temperature difference between the ice and the environment, and inversely proportional to the ambient density gradient. This result is similar to the laboratory and theoretical findings of Thorpe, Hutt and Soulsby (1969) for layers produced in a salt stratification by heating a sidewall. The difference is that in the case of a heated sidewall, the lateral salt flux is zero at the wall, while in the case of melting ice the freshwater flux and the (negative) heat flux are linked. In the laboratory experiments, and presumably in the field, such intrusions appear to be driven by diffusive interface fluxes, and rise upwards from the cold surface as they spread outwards. This causes difficulty in retrieving fresh water from melting icebergs towed to low latitudes to provide drinking water.

Laboratory experiments with melting ice-blocks in unstratified seawater have shown a cold, fresh convecting, doubly-diffusive boundary layer adjacent to the ice surface, with complicated physics and opposing buoyancy effects due to the combined action of freshwater and latent heat of cooling (Josberger and Martin, 1981). Similar boundary layers occur in the stratified experiments above, and likely in the field, and are undoubtedly involved in the formation and driving of intrusions. Kerr (1991) has suggested approximate boundary conditions for this situation. However, intrusive layer formation due to melting iceblocks has not to our knowledge been studied theoretically.

2.3 Effects of rotation and baroclinic shear

Eddy diffusivity parameterisations

The earliest theory of intrusions in a sheared, rotating flow is not thermohaline, but is double-diffusive in nature. McIntyre (1970) considered the stability of a baroclinic shear flow in either rectangular or circularly symmetric geometry. Diffusion of momentum and density was parameterised by eddy coefficients with ratio equal to the turbulent Prandtl number, σ. Density was considered as a single component, equivalent to equal eddy diffusivities for heat and salt. McIntyre found both steadily growing and oscillatory modes with slopes that depend on σ and the Richardson number of the shear. The mechanism of instability involves differential diffusion of mass and momentum (and indeed requires $\sigma \neq 1$), using either the baroclinic potential energy or the kinetic energy of the shear flow. The mechanism is discussed by Ruddick (1993) in a parallel manner to double-diffusive thermohaline intrusions following the fluid parcel arguments of Charney (1973).

Toole and Georgi (1981) included the effects of planetary rotation but not baroclinic shear in their theory of thermohaline intrusions, and also found an overstable mode - one which has short-term physics similar to an inertial oscillation, but with growing amplitude due to double-diffusion.

McDougall (1985a) considered the possibility of intrusive growth as "slab" layers using the formulation of Toole and Georgi but assuming uniform layer properties with interfacial flux laws. This has the advantage of reducing the PDE system to a system of ODE's and allowing the vertical fluxes to be specified in terms of layer T and S differences following laboratory "4/3" flux laws. This demonstrated a direct equivalence to the Toole-Georgi theory, with squared vertical wavenumber m^2 in Toole and Georgi (1981) becoming m in McDougall (1985a). McDougall (1985a) showed that, in the absence of mesoscale shear, the fastest-growing layers have a significant along-front tilt sufficient to balance Coriolis forces and allow the intrusive velocities to remain in the cross-frontal direction. Additionally, the growth rate and slope of the fastest-growing modes are not affected by Coriolis effects.

Molecular diffusivities

A study of intrusions using molecular diffusivities corresponding to the Toole and Georgi (1981) configuration was carried out by Kerr and Holyer (1986). This extended the work of Holyer (1983) by adding the effect of rotation about a vertical axis. Kerr and Holyer (1986) derived the same conclusion as McDougall (1985a): the growth rate of the fastest growing mode is not affected by rotation about a vertical axis, nor is the slope of the intrusions in the direction of the horizontal gradients, but that these intrusions do develop a slope in the perpendicular horizontal direction. A study by Worthem, Mollo-Christensen and Ostapoff (1983) also included the effects of vertical rotation on intrusions driven by infinite uniform gradients, but also included horizontal rotation and shear (which lead to horizontal density gradients). Because of the shear the modes of instability with along front slope were excluded, and so the effect of rotation was to reduce the growth rate of the fastest growing mode.

Finite width fronts

How rotation about a vertical axis affects oceanic fronts of finite width was investigated by Yoshida, Nagashima and Niino (1989). They discovered that rotation enhanced the instability of the front to intrusions as well as introducing a slope to the intrusive layers along the front. When the rotation rate was increased plots of the growth rate as a function of the wavenumbers of the instabilities show a second maximum developing which for sufficiently high rotation rates becomes the most unstable mode. This new mode of instability has a much larger vertical scale. Again, almost identical behaviour is found for an equivalent problem where the effect of rotation on convection in a salt stratified fluid between differentially heated vertical boundaries was investigated (Kerr 1995). There it was shown how these new rotational modes consist of fluid moving in almost horizontal closed paths which are driven by the vortex stretching and compression caused by the small vertical motions in the vertically adjacent convection cells. This is the same mechanism for the instabilities found using the oceanic parameterisation, but again the understanding of the mechanism came through a study using molecular diffusivities. An earlier theoretical study using molecular diffusivities by Chereskin and Linden (1986) assumed that the instabilities did not develop any slope parallel to the walls of the slot. In this case the instabilities were stabilized by the rotation. This assumption was appropriate for many of their experiments. They looked at intrusions growing from a central heated circular cylinder. For a given height of intrusions the slope parallel to the wall can only take discrete values, and so for weaker rotation rates it is appropriate to assume no slope in this direction. For larger rotation rates the intrusions in their experiments appear to no longer be axisymmetric.

Kuzmina and Rodionov (1992), and May and Kelley (1997) investigated the possibility of baroclinic, double-diffusive intrusion growth in what amounts to a triple-diffusive system - one with rotation, shear, and lateral T-S gradients. They consider the case of low and high shear, with and without alongfront slope, and delineate conditions under which the growth rate may be enhanced or diminished by baroclinic energy release. In addition, two new interleaving modes are predicted by May and Kelley (1997) that are mixed double-diffusive and baroclinic instabilities.

Equatorial intrusions

How important double diffusive processes are for the formation and maintenance of intrusions near the equator is as yet an open question. Richards (1991) shows that the observed vertical and horizontal scales are consistent with linear double diffusive layering instability on an equatorial beta plane. However, the later work of Edwards and Richards (1999) demonstrates that inertial instability applied to the equatorial ocean also has very similar characteristics to the observations making it difficult to discriminate between the two formation mechanisms. Edwards and Richards (1999) develop a theory of combined double-diffusive-inertial instability. In this case a third class of instability is found which has an oscillatory behaviour. The oscillatory behaviour can be attributed to differential diffusion of density and momentum, as discussed by McIntyre (1970).

3. Finite-amplitude Theories

Linear intrusion theories predict fastest-growing scales and slopes, but also give no information on the secondary instabilities or processes that guide the evolution of and serve to limit intrusion growth at finite amplitude. Since typical growth rates (O(day)) are much shorter than the lifetime of fronts, intrusions may spend most of their lifetime in a finite-amplitude state, and so fluxes are probably limited by the processes that control intrusion amplitude.

We know that at the smallest scales, fluxes are governed by molecular diffusivities, so theories like that of Holyer (1983) should apply. However, it is likely that in fact several instabilities occur in sequence, beginning with linear salt fingers, followed by nonlinear flux-limiting instabilities that govern finger fluxes, followed by linear intrusive instabilities, followed by one or more nonlinear processes that govern the evolution of thermohaline intrusions. As the intrusion amplitude becomes large, inversions in T and S can form, allowing the formation of diffusive-sense thermohaline convection above the warm, saline intrusions. Such a cascade of instabilities is difficult to understand or model through all the important stages, and is currently impossible to model numerically. Hence, many finite-amplitude theories attempt to parameterise the double-diffusive fluxes, and investigate the nonlinear behaviour of the resulting intrusions.

Joyce (1977) develops a model that uses conservation of heat and the observed intrusive finestructure gradient variance to estimate lateral heat flux in terms of an assumed vertical diffusivity. He makes no dynamical assumptions, and does not even assume double-diffusive mixing, although the need to assume a value of the vertical diffusivity is a major difficulty. The model has been applied in many observational situations, resulting in lateral diffusivities ranging from O(1) to O(1000) m^2s^{-1}. Application of the Joyce (1977) model requires finescale observations, and so cannot be used to predict lateral diffusivites in situations where the intrusive finestructure is unknown, such as in general circulation models.

Ruddick and Turner (1979) found in a series of laboratory experiments that the intrusion scale was proportional to the cross-frontal salinity contrast. They set out a mechanistic argument for the redistribution of heat and salt by intrusive advection followed by salt finger rundown, and considered the potential energies involved. They were able to deduce bounds for the vertical intrusion scales that were supported by the laboratory observations, and consistent with a variety of published ocean observations. The resulting estimate for the intrusion scale is:

$$H = \frac{3}{2}\frac{(1-\gamma)g\beta\Delta S}{N^2} \tag{7.}$$

Where N is the buoyancy frequency, and $\beta\Delta S$ is the density contrast due to the cross-frontal salinity change, measured along isopycnals (and therefore equal to the density contrast due to the cross-frontal temperature change). This form is similar to that found by Thorpe et al. (1969), Chen et al. (1971) and others for heated sidewall in a salt stratification, and Huppert and Turner (1980) for melting ice in salt stratified water. However, McDougall (1986) used arguments on the T–S plane similar to those of Ruddick and Turner (1979) to show that the evolution postulated by Ruddick and Turner predicts *T–S* structures inconsistent with oceanic observations. The neglect of the diffusive interface by Ruddick and Turner may explain the problems noted by McDougall.

McDougall, (1985b), considered the finite-amplitude evolution of the "slab" model of McDougall (1985a), and showed how inversions form to create a diffusive interface, which enters the flux balance significantly. McDougall (1985b) found that a steady three-way balance can occur involving lateral advection, finger fluxes, and diffusive fluxes, and argued for such a finite-amplitude steady state. The evolution to a similar state in a one-dimensional numerical model is illustrated in two figures below, from Walsh and Ruddick (1998), a numerical study of intrusion dynamics.

Figure 2 shows the S, T, velocity, density, and density ratio profiles in one experiment, at dimensionless times 0 (top row), 6000 (middle row), and 16800 (bottom row). The initial state is unstable to salt fingering, with a density ratio of 1.6. Sinusoidal perturbations develop and grow exponentially, as shown in figure 1, until inversions in salinity form as shown in the middle row. At this time, regions that are stable in T and S form and grow until inversions in T allow the formation of diffusive convection. The quasi-steady state at long time (bottom row) shows alternating finger and diffusive regions, each separated by convecting zones (positive density gradient).

Figure 3 shows the evolution of the double-diffusive regime versus dimensionless time and depth. The appearance of S-inversions at t=6000 marks the beginning of the formation of stable regions. A short time later, the amplitude grows enough to allow a T-inversion and a diffusive region to form, followed by the appearance of convecting zones at t=7000, and the adjustment of this state to equilibrium.

Merryfield (2000) considers the growth of intrusions to finite amplitude, and deduced conditions under which intrusions were unlikely to grow to the point of forming inversions. These "half-grown" intrusions are argued to remain as vertically periodic weakly intrusive structure, giving a possible mechanism for the formation of thermohaline staircases.

Ruddick, Phillips and Turner (1999) present a detailed analysis of the laboratory experiments initially described by Ruddick and Turner (1979), and conclude that the lateral fluxes can be deduced from observations of the "nose" velocity, c, of the intrusions, and that the nose velocity is well-fit by the scaling relation:

$$c = 0.005\, NH \qquad (8.)$$

They develop a kinematic theory, based on the heat and salt conservation equations, that considers the lateral structure and incorporates the intrusive noses. They justify the scaling relation for the nose velocity in an ad-hoc fashion, while noting that a complete analysis incorporating the momentum equations would be necessary to completely justify it. Their solution suggests that a similarity form solution is feasible.

In the previous sections comparisons between theories based on flux laws appropriate for the oceans and those based on molecular diffusivities show very strong parallels. However, when it comes to nonlinear theory we find that there are some results that have been obtained in one field that do not have analogues in the other. Sometimes this is because no clear parallel exists from one to the other, while at other times it is because the equivalent theory has not been attempted or is inappropriate.

One case where there possibly is a parallel is in the investigation of the onset of secondary instabilities in growing intrusions driven by molecular diffusivities (Kerr 1992). Here it was shown that growing intrusions could develop a secondary instability which are essentially intrusions growing in the core of the primary intrusions. These could lead to the mixing of the core of the intrusions. However, this mechanism is only important in the stage between the initial onset of instabilities and the establishment of fully developed intrusions, an intermediate stage that has not been observed in the oceans and may not be important there. Conversely, theories involving the modification of the vertical flux laws due to the modified vertical temperature and salinity gradients and the shear caused by the intrusions do not have a parallel in the study of intrusions driven by molecular diffusivities. There is clearly a parallel in the behaviour of fully developed intrusions that may be possible to exploit in the study of some fully developed intrusions driven by heating a salinity gradient from a sidewall, but this has not been done. Another area where there have been nonlinear results derived for intrusions driven by molecular diffusivities, but no equivalent for those driven by eddy fluxes is the weakly nonlinear analyses of Hart (1973), Kerr (1990) and Young and Rosner (1998). They showed that the instabilities in vertical slots and at heated sidewalls predicted by linear theory were all subcritical and so small amplitude intrusions would not be observed when the lateral gradients were only just greater than that required to initiate intrusion, but instead the intrusion would be expected to evolve to a large amplitude. These analyses showed that there was a tendency for the counter-rotating convection cells predicted by linear theory to evolve in such a way that the layers with fluid rising at the hot walls and sinking at the cool walls would grow, while the intrusions with circulation in the opposite sense would fade away. There cannot be any equivalent theory based around the Stern flux laws for ocean fronts as intrusions always have a finite growth rate and so they are never marginally unstable. For similar reasons there can be no equivalent of the energy stability analysis of Kerr (1990) who showed under what conditions arbitrary intrusions would be expected to decay for molecular diffusivities.

4. Remaining questions

4.1 The cornerstone

What are appropriate *vertical gradient* flux laws for a vertical double-diffusive stratification in which staircases have not formed (i.e., the gradient or "irregular steppy" situation)? Do they depend on the background level of turbulence, shear, or finescale internal waves? Once this difficult question is answered, it will be straightforward to model intrusive situations numerically. So far, it has been argued on dimensional grounds (Kelley, 1984) and assumed by those attempting to model the effects of double-diffusive processes, that the diffusivities are primarily a function of density ratio, with salt/heat flux ratio that is similar or equal to that found in the laboratory . Kelley (1984, 1986, 1988, 1990) uses dimensional arguments to collapse observational data on diffusive layer scales and develop gradient diffusivity laws for diffusive stratification. Although similar functional forms have been assumed for salt fingers (Schmitt, 1981), observational data on layer thicknesses have not had similar success (Schmitt, 1994).

Despite our lack of detailed knowledge of vertical gradient double-diffusive flux laws, the Stern (1967) flux ratio formulation and presumed density ratio dependence has been used to model intrusions theoretically (Walsh and Ruddick, 1995a,b) and numerically (Walsh and Ruddick, 1998, 2000), as well as to investigate the effects of vertical double-diffusion on water mass properties and transports in general circulation models (Merryfield et al, 1999, Zhang et al 1999). The use of such flux formulations in circulation models exhibiting thermoclinicity raises the question: can lateral intrusions form in such models? Although there is no obvious reason why not, it is likely that the fastest-growing intrusions have vertical scales smaller than the vertical model resolution. The effects of such sub-grid scale instabilities has not been investigated.

4.2 Dynamical Questions

A major dynamical question that is still unanswered is: "do linear theories adequately explain the intrusions we observe?" While a number of intrusion observations have been reported, there has been no systematic study designed to address that question. For example, do the slope and wavelength of intrusions always match those of the fastest-growing intrusions? Such questions are difficult to answer because of the lack double-diffusive gradient-based flux laws, discussed above. However, a great deal is now known about double-diffusive fluxes, and closer, more systematic comparison between observations and theory is feasible.

What do the Coriolis effect, and baroclinic velocity shear do to continuously stratified intrusions? Such questions have been addressed for linearized growing intrusions (Worthem, Mollo-Christenson and Ostapoff, 1983, Yoshida, Nagashima and Niino, 1989, Kuzmina and Rodionov, 1992, May and Kelley, 1997), and in a laboratory experiment (Chereskin and Linden, 1986 - heated sidewall, no shear), but the effects on finite-amplitude steady state intrusions have not been investigated. Such models predict along-front slopes for intrusions, reminiscent of the geostrophic balance, but Carmack et al. (1998) find along-front slopes to be much smaller than expected. May and Kelley (1997) hypothesize that vertical along-front shear can suppress the slope, but much work remains.

What is the importance and effect of the vertical recirculation within intrusions? Ruddick et al (1999) observed that the horizontal velocity in laboratory intrusions exceeded the nose velocity by a factor of about 3.7, and this implies a significant vertical recirculation within the intrusions. What is the effect of such a vertical velocity on the finger fluxes?

What is the role of the "noses"? In other words, intrusions must end somewhere, and in the laboratory they end in nose-like features reminiscent of the nose of a collapsing wake or mixed region. Do they serve simply to connect the intrusive regions, with double-diffusive fluxes, to non-intrusive regions, with sheared horizontal velocities? Or do noses play a much more significant role in determining the speed of advance of intrusions? Are "noses" like the one noted by Marmorino (1991) in the C-SALT staircase involved in the maintenance of the T-S signature of layers in thermohaline staircases? Are intrusive dynamics involved in the formation of layers (Merryfield, 2000)?

We have seen that there are many areas in the theory of intrusions where there are parallels between results derived using ocean flux laws and those using molecular diffusivities, and some areas where there is theory using only one of these. There are also other areas where theory is lacking for both approaches. For example what determines the stability of intrusions? In the laboratory and in numerical simulations intrusions growing from a heated wall are sometimes observed to merge as they evolve, but later on seem relatively stable. Is there a parallel with the apparent long time and distance stability of intrusions in the oceans? The dynamics of the fronts of intrusions is not well understood quantitatively. The ultimate distance that intrusions will propagate into stratified water from a heated sidewall is not known. A similar problem exists in the oceans: at a front between two converging bodies of water how far would intrusions extend into the oncoming flows? The answer to any of these questions using either an analysis involving molecular diffusivities or salt-finger fluxes will almost certainly give a strong indication of the results using the other approach. The two approaches often have strong parallels and should not be considered in isolation.

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Figures

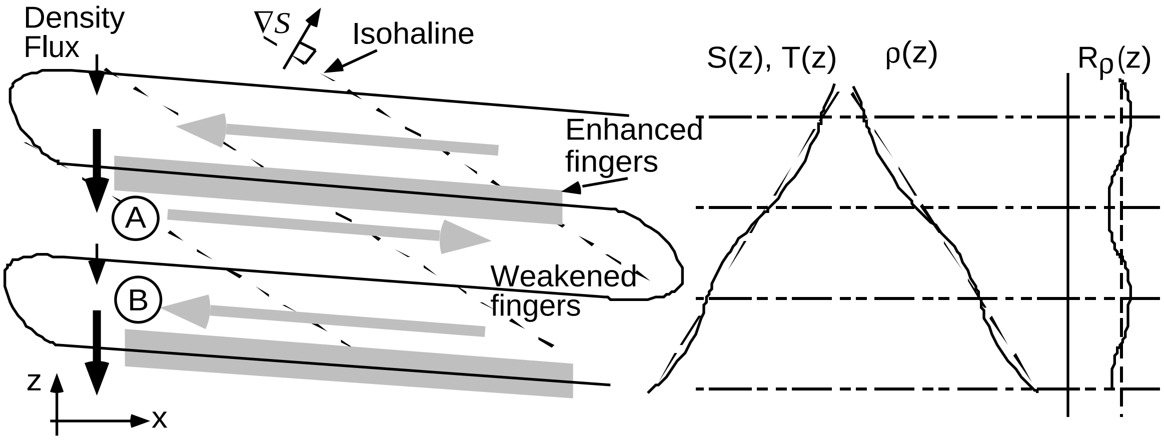

Figure 1. Linear phase of intrusion growth, with warm, saline water above and to the right. Isohaline surfaces (dashed lines) slope upwards to the left. Intrusive perturbations (dashed lines) also slope upwards to the left, but less steeply. Intrusive velocities are indicated by the gray arrows. Lateral advection by the intrusive velocities has caused the salinity gradients to be enhanced and density ratio to decrease in the hatched regions, resulting in increased finger fluxes. In between (dashed lines), the opposite occurs, causing decreased finger fluxes. The flux convergences cause fluid parcel a to become less dense, and fluid parcel b to become more dense. These density perturbations combine with the intrusion slope to reinforce the original motion. The vertical profiles of S, T, density, and density ratio are shown schematically on the right.

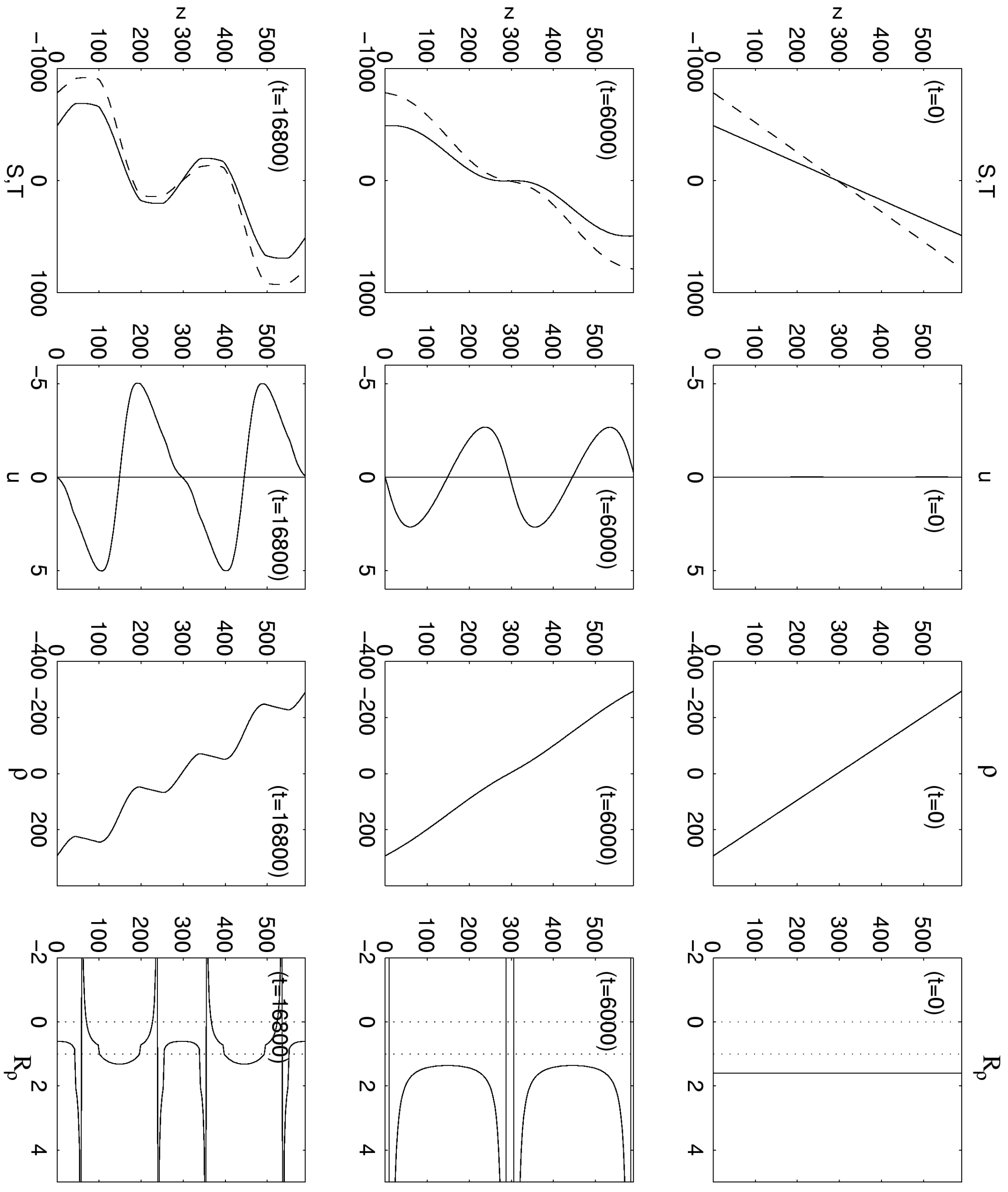

Figure 2. (Reproduced by permission of the American Meteorological Society, from Walsh and Ruddick, 1998, figure 7.) Plots of S (solid line), T (dashed line), velocity, density, and density ratio at three different times. (Note the figure appears on its side here -- apologies to the reader)

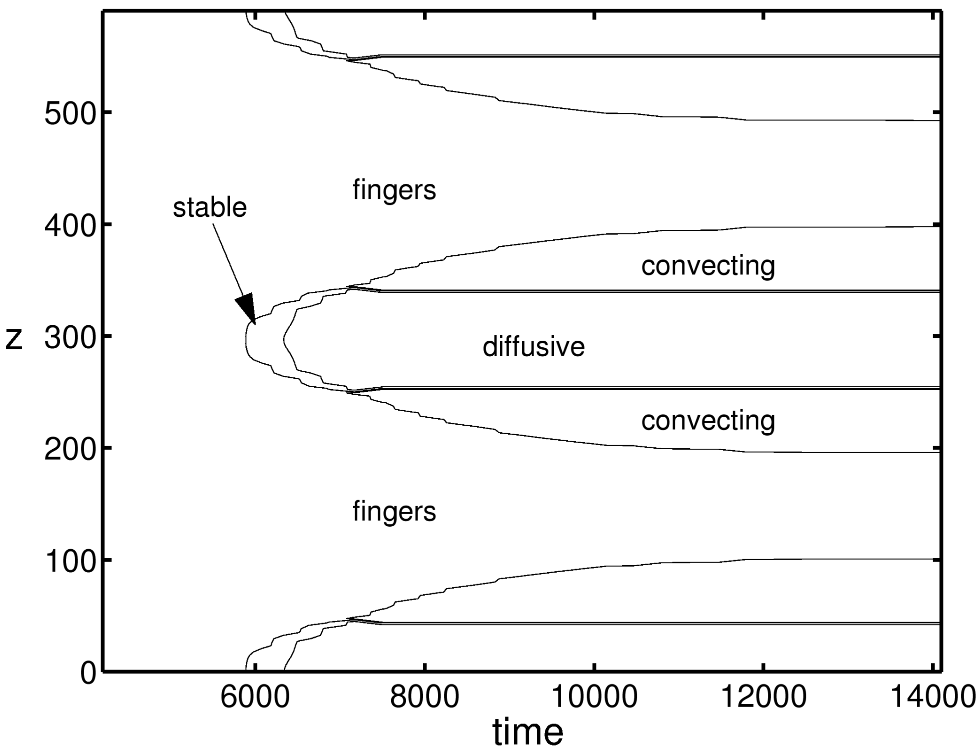

Figure 3. (Reproduced by permission of the American Meteorological Society, from Walsh and Ruddick, 1998, figure 5.) Evolution of layer thicknes and double-diffusive/convective regime for the model run shown in figure 2.