A Note on Centred and Upstream Numerical Advection Schemes

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Abstract: I examine two numerical solutions to the one-dimensional tracer advection equation, $\partial_t C + u\partial_x C = 0$, where C is a scalar concentration, u is a velocity, t is time, and x is the spatial coordinate. The algorithms examined are fully explicit. One uses a centred advection scheme while the other uses an upstream advection scheme. Results of the two schemes, and effects of boundary conditions on long-term evolution are discussed briefly. It is concluded that while the upstream scheme the is better of these two, one with less numerical diffusion would be preferable.

1 Introduction

The choice of algorithm used to examine a given differential equation numerically affects the quality and utility of the solution. Here, I have examined two possible schemes that can be used to solve the one-dimensional tracer advection equation,

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = 0 \tag{1}$$

where C is a some scalar, u is a velocity, t is time, and x is the spatial coordinate. The schemes are (i) centred time with centred advection and (ii) forward in time with upstream advection, and the discretizations for each are

(i) centred advection scheme:

$$C_j^{n+1} = -2u\Delta t\frac{C_{j+1}^n - C_{j-1}^n}{2\Delta x} + C_j^{n-1} \tag{2}$$

(ii) upstream advection scheme:

$$C_j^{n+1} = -u\Delta t\frac{C_j^n - C_{j-1}^n}{\Delta x} + C_j^n \tag{3}$$

where the superscripts and subscripts show discretization in time and space, respectively.

A brief overview of the domain is given in the following section. Output from of the two advection schemes are shown and discussed in Section 3. Interplay the centred scheme with boundary conditions is examined in Section 4. A brief conclusion is given in Section 5.

2 Methods

The model domain is described in Table 1. The model was run for 800 timesteps ($n_{max} = 800$, $t_{max} = 800$ s). The initial scalar concentration is described by the square function,

$$C_j^1 = \begin{cases} 0 & j < 2 \\ 2 & 2 \leq j < 20 \\ 0 & j \geq 20 \end{cases} \tag{4}$$

Given the domain size ($L = 1000$ m) and the velocity ($u = 0.2$ m s^{-1}), the square function will still be well within the domain at the end of the iteration. Hence, the choice of boundary conditions should not affect the long term evolution for this problem so long as they are stable. I have chosen the boundary condition $\partial C/\partial x = 0$, although clamped boundary condtions could also be considered as valid. The boundary conditions (and choice of scheme) can affect the solution when the concentration spike is advected out of the model domain through an open boundary. This scenario is examined in the Section 4.

Variable	Symbol	Value
time step	Δt	1 s
space step	Δx	1 m
velocity	u	0.2 m s^{-1}
domain length	L	1000 m
duration	t_{max}	800 s

Table 1: Parameters defining the model domain.

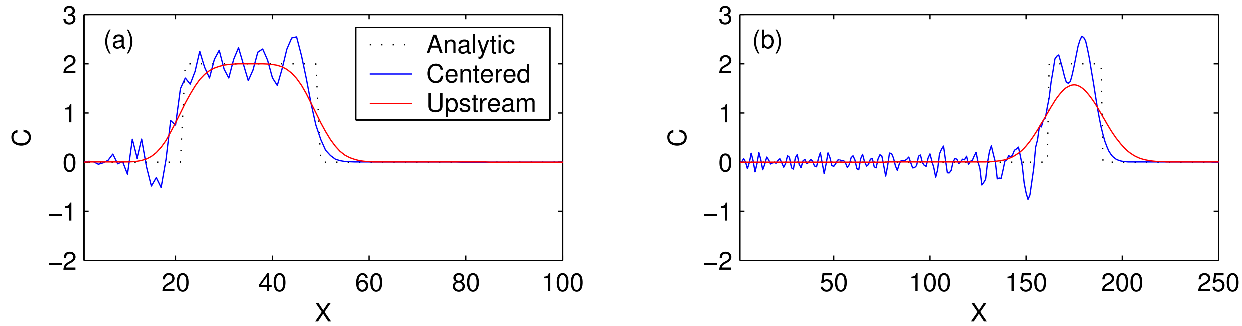

Figure 1: Results of two numerical schemes after (a) 100 s and (b) 800 s.

3 Results

Results from the two advection schemes are shown in Figure 1.

(i) centred advection scheme:

The centred advection scheme keeps the overall shape of the analytic solution as it evolves in time. It reports negative tracer concentrations, which are spurious. There is also a large amount of high-wavenumber noise upstream of the concentration "spike". Neither of these features is desireable. Over the full 800 seconds the total concentration, i.e. $\sum_j C_j$, increases by 0.03%.

(ii) upstream advection scheme:

In contrast to the centred advection scheme, the upstream scheme does not have the same variability upstream of the concentration spike, nor does it produce spurious negative concentrations. It does, however, have a high amount of numerical diffusion. It would be better to utilize an advection scheme that limited this "leakage". Conservation of C is better for this scheme than the centred scheme: $\sum_j C_j$ decreases only by O(10^{-15}) over the 800 seconds of the numerical trial ($\epsilon = 2 \times 10^{-16}$).

4 Boundary Considerations

A second trial, with $L_2 = 100$ m was also examined to consider the effects of boundaries on the advection schemes (results are shown in Figure 2). When the tracer spike encounters an open boundary, the centred advection scheme shows a very undesirable result: a spike of "negative concentration" is reflected back into the domain. Rather than having the spike advect out of the domain, as is the case for the upstream advection scheme, a signal gets trapped in the domain. If the model is allowed to continue running (by increasing t_{max}), the solution eventually goes unstable and "blows up".

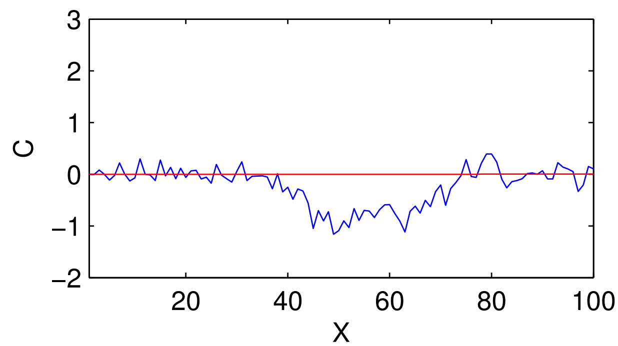

Figure 2: Same as Figure 1b but with a smaller domain, $L_2 = 100$ m.

5 Conclusion

It is evident that the upstream scheme is the better of the two schemes discussed. It does not produce spurious negative concentrations, and it behaves well with the open boundaries. This scheme, however, has problems with numerical diffusion. Other schemes, such as one of higher order with a flux limiter, likely provide a better alternative.