In Preparation

An Overview of Jacobian Elliptical Functions

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Abstract:

1 Introduction

To begin, let us consider the function

$$f(t) = \frac{1}{\sqrt{1-t^2}}. \tag{1}$$

In introductory calculus courses, one learns early on that

$$F(x) = \int_0^x f(t)dt = \sin^{-1}(x). \tag{2}$$

The function $F(x)$ has real values for $-1 \leq x \leq 1$, and is periodic with a period $T = 2\pi$. The principal values of $F(x)$ are bound by $-\pi/2 \leq F(x) \leq \pi/2$. If we define K such that

$$K = \int_0^1 f(t)dt, \tag{3}$$

then $K = \pi/2$, and the period of $F(x) = 4K$. Because of this relationship, K is often called the *quarter period*. The function, $F(x)$, known as a cyclometric function, is simply the inverse of the elementary function, $\sin\phi$, i.e. $F(\sin\phi) = \phi$.

We will now move onto a more complex function that will form the basis for Jacobian Elliptical Functions, or JEFs. Consider now the function

$$G(\phi, k) = u = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}. \tag{4}$$

If we now make the substitutions $t = \sin\theta$ and

$$x = \sin\phi, \tag{5}$$

then we may rewrite this function as

$$u = \int_0^x g(k,t)dt = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{6}$$

For $k = 0$, $g(k,t)$ reduces to $f(t)$ and $u = \sin^{-1} x$, or $x = \sin u$. For $k = 1$, $u = \tanh^{-1} x$, or $x = \tanh u$.

2 Defining Jacobian Elliptical Functions

We can now define our first JEF, sn, as the inversion $u(x,k)$ with respect to x, namely

$$\mathrm{sn}(u,k) = x = \sin\phi. \tag{7}$$

Other notations may write function as $\mathrm{sn}(u|k)$ or $\mathrm{sn}(u)$, where the k is assumed but not included.

In an analogy to the trigonometric relationship between $\sin(x)$ and $\cos(x)$, there exists a function $\mathrm{cn}(u,k)$,

$$\mathrm{cn}(u,k) = \sqrt{1-\mathrm{sn}^2(u,k)} = \sqrt{1-x^2} = \cos\phi. \tag{8}$$

The ratio $\mathrm{sn}(u,k)/\mathrm{cn}(u,k)$ was once called $\mathrm{tn}(u,k)$, but for reasons explained below is now referred to as $\mathrm{sc}(u,k)$,

$$\mathrm{sc}(u,k) = \frac{\mathrm{sn}(u,k)}{\mathrm{cn}(u,k)} = \frac{x}{\sqrt{1-x^2}} = \tan\phi. \tag{9}$$

A final JEF that compliments sn and cn, is

$$\mathrm{dn}(u,k) = \sqrt{1-k^2x^2} = (1-k^2\sin^2\phi)^{1/2} = \Delta(\phi). \tag{10}$$

In the limit that $k \to 1$, $\mathrm{dn} \sim \mathrm{cn}$. In the limit that $k \to 0$, these functions tend towards their trigonometric counterparts, except for dn which tends towards unity. A graphical display of these functions for $k = 1/2$ is shown in Figure 1.

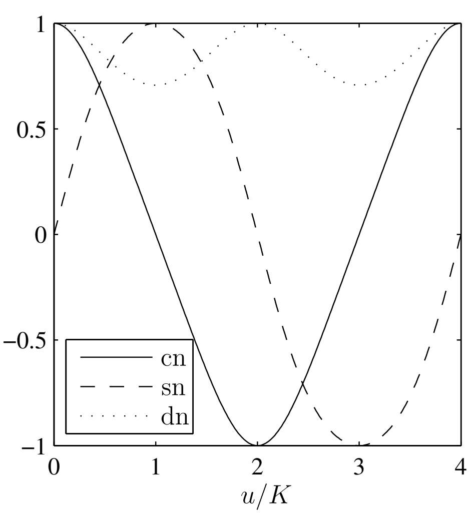

Figure 1: Jacobian elliptical functions cn, sn, and dn for $k = 1/2$. Note that the period for $\mathrm{dn}(u,k)$ is only $T = 2K(k)$, as opposed to $T = 4K(k)$ for sn and cn. For $k > 1/2$, cn curves have an inflection point at $u/K = 1$ and $u/K = 3$.

3 The Quarter Periods

Further understanding of the JEFs may be gained by returning to the quarter period, K. We expand the function given above in (3) so it carries a dependence on k,

$$K(k) = \int_0^1 f(k,t)dt = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad (11)$$

noting that $4K(0) = 2\pi$ and $4K(1) = \infty$ suggests that the period of $\mathrm{sn}(u,k)$ is $T = 4K(k)$. The quantity k is called *the parameter* by Abramowitz and Stegun (1965), and a *nonlinearity parameter* by Apel (2003). This latter definition arises from the concept that as $k \to 1$, $\mathrm{cn}^2(u,k) \sim \mathrm{sech}^2(u)$, i.e. the solitary wave solution of the KdV equation (Korteweg and de Vries 1895). For now, we will use the *parameter*. The function K is shown in Figure 2.

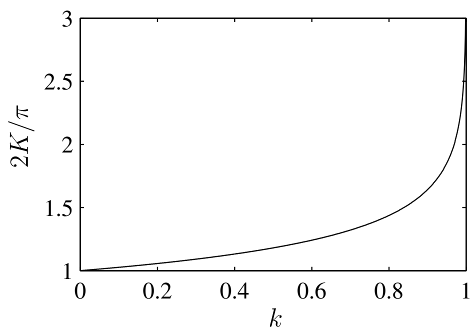

Figure 2: The function $K(k)$. Note that $K \sim \infty$ as $k \to 1$.

The parameter is joined by a *complimentary parameter* k_1 such that

$$k + k_1 = 1 \quad (12)$$

and, by extension,

$$K'(k) = K(k_1). \quad (13)$$

If one of k, k_1, K, K', or K'/K are known, then the rest may be determined (Abramowitz and Stegun 1965).

A graphical relationship between the quarter period and the JEFs can be constructed using an Argand diagram (Figure 3). If we take p and q as any two of the letters c, d, n, and s, then the function $\mathrm{pq}(u,k)$ must have the following three properties (Abramowitz and Stegun 1965):

1. $\mathrm{pq}(u,k)$ has a simple zero at p and a pole at q.

2. the step from p to q is a half-period of $\mathrm{pq}(u)$. Also, $\mathrm{pq}(u,k)$ is periodic in the other two directions (i.e. from p to one of the other two corners of the rectangle), with a period such that the distance from p to the other corners is a quarter period.

3. The coefficient of the leading term in the expansion of $\mathrm{pq}(u,j)$ in ascending powers of u about $u = 0$ is unity. If the expansion is at p then the leading power is u^1, if it is at q then the leading power is u^{-1}, and if it is at one of the other two corners, then the leading power is u^0.

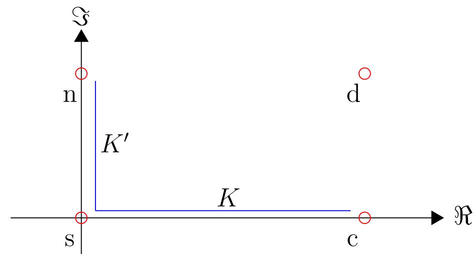

Figure 3: Argand diagram showing the relationship between the quarter period and JEFs.

A first glance at statements 1 and 2 may not seem to match with Figure 1. However, in the case here, u and K are real, and as such Figure 1 is a projection of these complex functions onto the real axis.

4 Relationships

All relationships transcribed from Abramowitz and Stegun (1965).

4.1 Ratios

If p, q, and r are any three letters of c, d, s, and n, then

$$\mathrm{pq}(u,k) = \frac{\mathrm{pr}(u,k)}{\mathrm{qr}(u,k)} \quad (14)$$

4.2 Squares

$$-\mathrm{dn}^2(u,k) + k_1 = -k\,\mathrm{cn}^2(u,k) = k\,\mathrm{sn}^2(u,k) - k \quad (15)$$

$$-k_1\,\mathrm{nd}^2(u,k) + k_1 = -kk_1\,\mathrm{sd}^2(u,k) = k\,\mathrm{cd}^2(u,k) - k \quad (16)$$

$$k_1\,\mathrm{sc}^2(u,k) + k_1 = k_1\,\mathrm{nc}^2(u,k) = \mathrm{dc}^2(u,k) - k \quad (17)$$

$$\mathrm{cs}^2(u,k) + k_1 = \mathrm{ds}^2(u,k) = \mathrm{ns}^2(u,k) - k \quad (18)$$

4.3 Addition Theorems

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}(u)\,\mathrm{cn}(v)\,\mathrm{dn}(v) + \mathrm{sn}(v)\,\mathrm{cn}(u)\,\mathrm{dn}(u)}{1 - k\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)} \quad (19)$$

$$\mathrm{cn}(u+v) = \frac{\mathrm{cn}(u)\,\mathrm{cn}(v) - \mathrm{sn}(u)\,\mathrm{dn}(u)\,\mathrm{sn}(v)\,\mathrm{cn}(v)}{1 - k\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)} \quad (20)$$

$$\mathrm{dn}(u+v) = \frac{\mathrm{dn}(u)\,\mathrm{dn}(v) - k\,\mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{sn}(v)\,\mathrm{sn}(v)}{1 - k\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)} \quad (21)$$

4.4 Double Arguments

$$
\begin{aligned}
\mathrm{sn}(2u) &= \frac{2\,\mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{dn}(u)}{1 - k\,\mathrm{sn}^4(u)} \\
&= \frac{2\,\mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{dn}(u)}{\mathrm{sn}^2(u) + \mathrm{sn}^2(u)\,\mathrm{dn}^2(u)}
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
\mathrm{cn}(2u) &= \frac{\mathrm{cn}^2(u) - \mathrm{sn}^2(u)\,\mathrm{dn}^2(u)}{1 - k\,\mathrm{sn}^4(u)} \\
&= \frac{\mathrm{cn}^2(u) - \mathrm{sn}^2(u)\,\mathrm{dn}^2(u)}{\mathrm{sn}^2(u) + \mathrm{sn}^2(u)\,\mathrm{dn}^2(u)}
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
\mathrm{dn}(2u) &= \frac{\mathrm{dn}^2(u) - k\,\mathrm{sn}^2(u)\,\mathrm{cn}^2(u)}{1 - k\,\mathrm{sn}^4(u)} \\
&= \frac{\mathrm{dn}^2(u) + \mathrm{cn}^2(u)\left[\mathrm{dn}^2(u) - 1\right]}{\mathrm{dn}^2(u) - \mathrm{cn}^2(u)\left[\mathrm{dn}^2(u) - 1\right]}
\end{aligned} \tag{24}
$$

4.5 Half Arguments

$$\mathrm{sn}^2(\tfrac{1}{2}u) = \frac{1 - \mathrm{cn}(u)}{1 + \mathrm{dn}(u)} \tag{25}$$

$$\mathrm{cn}^2(\tfrac{1}{2}u) = \frac{\mathrm{cn}(u) + \mathrm{dn}(u)}{1 + \mathrm{dn}(u)} \tag{26}$$

$$\mathrm{dn}^2(\tfrac{1}{2}u) = \frac{k_1 + \mathrm{dn}(u) + k\,\mathrm{cn}(u)}{1 + \mathrm{dn}(u)} \tag{27}$$

5 Derivatives

$\mathrm{fn}(u,k)$	$\frac{d}{du}\,\mathrm{fn(u,k)}$
$\mathrm{cn}(u,k)$	$-\,\mathrm{sn}(u,k)\,\mathrm{dn}(u,k)$
$\mathrm{dn}(u,k)$	$-k\,\mathrm{sn}(u,k)\,\mathrm{cn}(u,k)$
$\mathrm{sn}(u,k)$	$\mathrm{cn}(u,k)\,\mathrm{dn}(u,k)$
$\mathrm{cd}(u,k)$	$k_1\,\mathrm{sd}(u,k)\,\mathrm{nd}(u,k)$
$\mathrm{nd}(u,k)$	$k\,\mathrm{sd}(u,k)\,\mathrm{cd}(u,k)$
$\mathrm{sd}(u,k)$	$\mathrm{cd}(u,k)\,\mathrm{nd}(u,k)$
$\mathrm{dc}(u,k)$	$k_1\,\mathrm{nc}(u,k)\,\mathrm{sc}(u,k)$
$\mathrm{nc}(u,k)$	$\mathrm{dc}(u,k)\,\mathrm{nc}(u,k)$
$\mathrm{sc}(u,k)$	$\mathrm{dc}(u,k)\,\mathrm{nc}(u,k)$
$\mathrm{cs}(u,k)$	$-\,\mathrm{ds}(u,k)\,\mathrm{ns}(u,k)$
$\mathrm{ds}(u,k)$	$-\,\mathrm{cs}(u,k)\,\mathrm{ns}(u,k)$
$\mathrm{ns}(u,k)$	$-\,\mathrm{cs}(u,k)\,\mathrm{ds}(u,k)$

Table 1: Derivatives of Jacobian Elliptical Functions, separated by poles. Not that the derivative is proportional to the two copolar functions.

6 Differential Equations

The functions $\mathrm{cn}(u,k)$, $\mathrm{dn}(u,k)$, and $\mathrm{sn}(u,k)$ are solutions to the differential equations

$$\frac{d^2 f}{dx^2} = -(1 + k^2) f + 2k^2 f^3 \tag{28}$$

$$\frac{d^2 f}{dx^2} = -(1 - 2k^2) f - 2k^2 f^3 \tag{29}$$

$$\frac{d^2 f}{dx^2} = (2 - k^2) f - 2 f^3 \tag{30}$$

Acknowledgments

References

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Korteweg, D. J. and de Vries, G.: 1895, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, *Phil. Mag.* **39**, 422–443.