Start with conservation of mass and momentum (neglecting friction, assuming Boussinesq):

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho} + \mathbf{g} \tag{1}$$

Assume irrotational flow, i.e. $\mathbf{u} = \nabla\phi$, so (1) can be written as

$$\nabla\left[\partial_t\phi + \frac{1}{2}(\nabla\phi\cdot\nabla\phi) + \frac{p}{\rho} - gz\right] = 0 \tag{2}$$

Incorporate constant of integration into ϕ, which is defined by

$$\partial_t\phi + \frac{1}{2}(\nabla\phi\cdot\nabla\phi) + \frac{p}{\rho} - gz = 0 \tag{3}$$

and (from irrotational assumption)

$$\nabla^2\phi = 0 \tag{4}$$

Boundary Conditions at Bottom

No flow through bottom: $\partial_z \phi \,|_{z=-h} = 0$

Boundary Conditions at Surface

Vertical flow:

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=0} = \frac{\partial \eta}{\partial t} + \left.\frac{\partial \phi}{\partial z}\right|_{z=0} \frac{\partial \eta}{\partial x} \qquad (5)$$

Horizontal flow (with no pressure gradient along surface):

$$\frac{\partial^2 \phi}{\partial x \partial t} + \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial z}\frac{\partial^2 \phi}{\partial x \partial z} + g\frac{\partial \eta}{\partial x} = 0 \qquad (6)$$

More Assumptions

Assume two dimensional ($\partial_y = 0$)

Make a long wave approximation ($\delta \equiv (h\ell^{-1})^2 \ll 1$),

Make a small amplitude approximation ($\epsilon \equiv ah^{-1} \ll 1$)

Assume ϵ and δ are of the same order ($\epsilon\delta^{-1} = O(1)$)

Expand in a power series in $\tilde{z} = z - h$:

$$\phi = \sum_{n=0}^{\infty} \tilde{z}^n \tilde{\phi}_n(x,t) \tag{7}$$

Laplace's equation (4) gives a recursion relation:

$$\sum_{n=0}^{\infty} \tilde{z}^n \left[\partial_{xx}\tilde{\phi}_n + (n+2)(n+1)\tilde{\phi}_{n+2}\right] = 0 \tag{8}$$

No flow through the boundary implies $\tilde{\phi}_1 = 0$, so all odd terms vanish.

Results of Expansion

$$\phi = \tilde{\phi}_o - \frac{1}{2}\tilde{z}^2 \partial_{xx}\tilde{\phi}_o + \dots \quad (9)$$

$$u = \phi_x = \partial_x \tilde{\phi}_o - \frac{1}{2}\tilde{z}^2 \partial_{xxx}\tilde{\phi}_o + \dots \quad (10)$$

$$w = \phi_z = -\tilde{z}\partial xx \tilde{\phi}_o + \frac{1}{6}\tilde{z}^3 \partial xxxx \tilde{\phi}_o + \dots \quad (11)$$

Non-Dimensionalization

$$x' = \frac{x}{\ell}, \quad z' = \frac{z}{h}, \quad t' = \frac{tc_o}{l}, \quad \eta' = \frac{\eta}{a}, \quad \phi' = \frac{\phi}{\epsilon \ell c_o}$$

(The surface is at $\epsilon\bar{\eta}$)

$$\phi' = \tilde{\phi}'_o - \frac{1}{2}(1+\epsilon\eta')^2 \frac{\partial^2 \tilde{\phi}'_o}{\partial x'^2} + O(\epsilon^2) \tag{12}$$

$$u' = \frac{\partial \tilde{\phi}'_o}{\partial x'} - \frac{1}{2}(1+\epsilon\eta')^2 \frac{\partial^3 \tilde{\phi}'_o}{\partial x'^3} + O(\epsilon^2) \tag{13}$$

$$w' = -\delta\left[(1+\epsilon\eta')\frac{\partial^2 \tilde{\phi}'_o}{\partial x'^2}\tilde{\phi}'_o + \frac{1}{6}\delta\frac{\partial^4 \tilde{\phi}'_o}{\partial x'^4}\right] + O(\epsilon^2) \tag{14}$$

At the Surface

If we introduce $u_o' = \partial_{x'}\tilde{\phi}_o'$, then after a little reworking (5) becomes (dropping the primed notation)

$$\frac{\partial \eta}{\partial t} + \epsilon u_o \frac{\partial \eta}{\partial x} + (1 + \epsilon\eta)\frac{\partial u_o}{\partial x} + \frac{\delta}{6}\frac{\partial^3 u_o}{\partial x^3} = 0 \qquad (15)$$

and (6) becomes

$$\frac{\partial u_o}{\partial t} - \frac{\delta}{2}\frac{\partial^3 u_o}{\partial x^2 \partial t} + \epsilon u_o \frac{\partial u_o}{\partial x} + \frac{\partial \eta}{\partial x} = 0 \qquad (16)$$

to $O(\epsilon^2)$.

At $O(1)$,

$$\partial_t \eta + \partial_x u_o = 0 = \partial_t u_o + \partial_x \eta \qquad (17)$$

so η and u_o are both solutions to the wave equation with a (dimensionless) wave speed of unity.

Use the ansatz* that η and u_o are similar at zeroth order,

$$u_o \equiv \eta + \epsilon\mathcal{F}(x,t) + \delta\mathcal{G}(x,t) + O(\epsilon^2) \tag{18}$$

and rewrite our surface boundary conditions in terms of η:

$$\frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial x} + \epsilon\left(\frac{\partial\mathcal{F}}{\partial t} + \eta\frac{\partial\eta}{\partial x}\right) + \delta\left(\frac{\partial\mathcal{G}}{\partial t} - \frac{1}{2}\frac{\partial^3\eta}{\partial x^2\partial t}\right) = 0 \tag{19}$$

$$\frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial x} + \epsilon\left(\frac{\partial\mathcal{F}}{\partial x} + 2\eta\frac{\partial\eta}{\partial x}\right) + \delta\left(\frac{\partial\mathcal{G}}{\partial x} - \frac{1}{6}\frac{\partial^3\eta}{\partial x^3}\right) = 0 \tag{20}$$

Subtraction gives

$$\epsilon\left(\frac{\partial\mathcal{F}}{\partial x} - \frac{\partial\mathcal{F}}{\partial t} + 2\eta\frac{\partial\eta}{\partial x}\right) + \delta\left(\frac{\partial\mathcal{G}}{\partial x} - \frac{\partial\mathcal{G}}{\partial t} - \frac{1}{2}\frac{\partial^3\eta}{\partial x^2\partial t} - \frac{1}{6}\frac{\partial^3\eta}{\partial x^3}\right) = 0 \tag{21}$$

*assumed form not based on an underlying theory

$$\epsilon\left(\frac{\partial\mathcal{F}}{\partial x}-\frac{\partial\mathcal{F}}{\partial t}+2\eta\frac{\partial\eta}{\partial x}\right)+\delta\left(\frac{\partial\mathcal{G}}{\partial x}-\frac{\partial\mathcal{G}}{\partial t}-\frac{1}{2}\frac{\partial^3\eta}{\partial x^2\partial t}-\frac{1}{6}\frac{\partial^3\eta}{\partial x^3}\right)=0$$

Expecting $\mathcal{F}$ and $\mathcal{G}$ to obey the wave equation with a (dimensionless) wave speed of unity gives

$$\frac{\partial\mathcal{F}}{\partial x}=-\frac{\partial\mathcal{F}}{\partial t},\qquad \frac{\partial\mathcal{G}}{\partial x}=-\frac{\partial\mathcal{G}}{\partial t}$$

and hence

$$\mathcal{F}=-\frac{1}{4}\eta^2,\qquad \mathcal{G}=-\frac{1}{3}\frac{\partial^2\eta}{\partial x^2}$$

and

$$u_o=\eta-\frac{\epsilon}{4}\eta^2+\frac{\delta}{3}\frac{\partial^2\eta}{\partial x^2}+O(\epsilon^2)\qquad(22)$$

From the end of the last slide,

$$u_o = \eta - \frac{\epsilon}{4}\eta^2 + \frac{\delta}{3}\frac{\partial^2 \eta}{\partial x^2} + O(\epsilon^2).$$

(15) is reproduced here:

$$\frac{\partial \eta}{\partial t} + \epsilon u_o \frac{\partial \eta}{\partial x} + (1 + \epsilon\eta)\frac{\partial u_o}{\partial x} + \frac{\delta}{6}\frac{\partial^3 u_o}{\partial x^3} = 0$$

Inserting our solution for u_o into (15) yields the KdV equation:

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} + \frac{3}{2}\epsilon\eta\frac{\partial \eta}{\partial x} + \frac{\delta}{6}\frac{\partial^3 \eta}{\partial x^3} = 0 \tag{23}$$