A Description of the Variational Principle introduced by

Turkington, Eydeland, and Wang, A Computational Method for Solitary Internal Waves in a Continuously Stratified Fluid, *Studies in Applied Mathematics*, **85**, 93-127.

Define a streamfunction, $\psi(x,y)$. In 2-dimensions, start with

$$\nabla \cdot \mathbf{u} = \nabla^2 \psi = 0 \tag{1}$$

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + J(\rho, \psi) = 0 \tag{2}$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g} \tag{3}$$

Now we introduce a density-weighted vorticity

$$\sigma = -\nabla \cdot (\rho \nabla \psi) \tag{4}$$

and taking $\nabla \cdot (3)$ yields

$$\frac{\partial \sigma}{\partial t} + J(\sigma, \psi) + J(\rho, gy - \tfrac{1}{2} \mid \nabla \psi \mid^2) = 0 \tag{5}$$

For a steady, translational feature, $\rho = \rho(x-ct, y), \sigma = \sigma(x-ct, y)$ and Equations (2) and (5) can be written as

$$J(\rho, \psi - cy) = 0 \tag{6}$$

$$J(\sigma, \psi - cy) + J(\rho, gy - \tfrac{1}{2} \mid \nabla\psi \mid^2) = 0 \tag{7}$$

Now, consider a density field

$$\rho(x, y) = \bar{\rho}(y - \eta(x, y)) \tag{8}$$

where $\eta(x, y)$ is the displacement of an isopycnal from its background state $\bar{\rho}(y)$ (i.e. $\eta \sim 0$ as $x \to \pm\infty$). Hence (6) can be written as

$$J(\bar{\rho}(y - \eta), \psi - cy) = \bar{\rho}'(y - \eta) J(y - \eta, \psi - cy) = 0 \tag{9}$$

It can readily be shown that

$$J(y-\eta,\psi-cy)=J(y-\eta,\psi-c\eta)=0 \tag{10}$$

and that the boundary conditions impose

$$\psi=c\eta. \tag{11}$$

The vorticity equation can be written as

$$J(y-\eta,c\sigma+\bar{\rho}'(y-\eta)[g\eta-\tfrac{1}{2}c^2\mid\nabla\eta\mid^2])=0 \tag{12}$$

which yields

$$\frac{\sigma}{c}-\tfrac{1}{2}\mid\nabla\eta\mid^2\bar{\rho}'(y-\eta)=-\frac{g}{c^2}\eta\bar{\rho}'(y-\eta). \tag{13}$$

Combining (11) and (13) gives the eigenvalue problem

$$\overbrace{-\nabla\cdot(\bar{\rho}(y-\eta)\nabla\eta)-\tfrac{1}{2}\mid\nabla\eta\mid^2\bar{\rho}'(y-\eta)}^{E'(\eta)}=\lambda\overbrace{\left[-\frac{\eta}{h}\bar{\rho}'(y-\eta)\right]}^{F'(\eta)} \tag{14}$$

$$\overbrace{-\nabla\cdot(\bar{\rho}(y-\eta)\nabla\eta)-\tfrac{1}{2}\mid\nabla\eta\mid^2\bar{\rho}'(y-\eta)}^{E'(\eta)}=\lambda\overbrace{\left[-\frac{\eta}{h}\bar{\rho}'(y-\eta)\right]}^{F'(\eta)}$$

where $\lambda = ghc^{-2}$ and h is the water depth.

$$E(\eta) = \int_D \tfrac{1}{2}\mid\nabla\eta\mid^2\bar{\rho}(y-\eta)dxdy \tag{15}$$

$$F(\eta) = \frac{1}{h}\int_D\left\{\int_0^\eta\left[\bar{\rho}(y-\eta)-\bar{\rho}(y-\xi)\right]d\xi\right\}dxdy \tag{16}$$

Note that the Kinetic Energy of the system is $c^2E(\eta)$, and that the Available Potential Energy of the system is $ghF(\eta)$.

$$\overbrace{-\nabla\cdot(\bar{\rho}(y-\eta)\nabla\eta)-\tfrac{1}{2}\mid\nabla\eta\mid^2\bar{\rho}'(y-\eta)}^{E'(\eta)}=\lambda\overbrace{\left[-\frac{\eta}{h}\bar{\rho}'(y-\eta)\right]}^{F'(\eta)}$$

Hence, given the appropriate boundary conditions, the structure of an arbitrary solitary internal wave can be found by solving (14) with the variational principle

$$E(\eta)\to\text{min}\qquad\text{subject to}\qquad F(\eta)=A\qquad(17)$$