The non-Boussinesq Temporal-Residual-Mean

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Abstract.

A method for modifying currently existing ocean models to make them non-Boussinesq has been advocated by McDougall, Greatbatch and Lu, and implemented in the POP model by Greatbatch et al.. Here, we provide theoretical justification for combining the above modifications with the Temporal Residual Mean (TRM) approach of McDougall and McIntosh. The TRM is a method for including the skew flux of tracers caused by the adiabatic stirring of mesoscale eddies in non-eddy-resolving ocean models. The paper provides the justification for simultaneously undertaking these two model improvements, and the physical interpretation of the model variables in this situation.

1. Introduction

The stirring by the mesoscale eddy field in the ocean is thought to occur primarily in the local, neutral tangent plane. Indeed, diapycnal transport (that is, transport across the local neutral tangent plane) is thought to be weak throughout the interior of the ocean, raising questions as to how the deep waters of the ocean are returned to the surface in order to close the thermohaline circulation (e.g. Munk and Wunsch, 1998; Webb and Suginohara, 2001). The accurate representation of mesoscale stirring is therefore an important issue in ocean modelling. In non-eddy permitting models, the mesoscale stirring must be parameterized, and the issue arises as to how the equations of motion should be averaged. To respect the fact that stirring occurs primarily along, rather than across the local neutral tangent plane, many studies have adopted isopycnal averaging [e.g. Gent et al., 1995; Dukowicz and Smith, 1997; Greatbatch, 1998; Dukowicz and Greatbatch, 1999; Smith, 1999]. Difficulties arise, however, at the ocean surface and bottom, where isopycnals intersect the boundaries in a highly, time-dependent fashion, and it is often not clear what boundary conditions are appropriate to isopycnally averaged quantities (e.g. Killworth, 2001). McDougall and McIntosh(2001; hereafter MM) address these difficulties by introducing the Temporal-Residual-Mean (TRM). The TRM provides a way to represent isopycnally-averaged equations and quantities in a height-coordinate framework. In MM, the horizontal component of the tracer transport velocity (the velocity by which the appropriately averaged tracer field is advected) is the thickness-weighted average at constant density of the instantaneous horizontal velocity. MM represent the eddy-induced part of the transport by a quasi-Stokes streamfunction for the difference between the tracer transport velocity and the Eulerian mean velocity averaged at fixed height. MM note that the behaviour of the quasi-Stokes streamfunction near the boundaries can easily be understood, and since the boundary conditions that must be applied to the Eulerian mean velocity are straightforward, the MM framework provides a potentially tractable approach to mesoscale eddy parameterization in ocean models, particularly the widely used height-coordinate models (e.g. the MOM code [Pacanowski and Griffies, 1999; Griffies et al., 2000]).

The treatment of the TRM in MM uses the Boussinesq equations of motion. Here we show how to generalise the TRM to a non-Boussinesq ocean using the approach of McDougall, Greatbatch and Lu(2002; hereafter MGL). The result is an elegant, unified treatment, combining the TRM approach to mesoscale eddy parameterization with the accurate treatment of the non-Boussinesq equations of motion introduced by MGL. The basic message of the paper is that to make ocean models fully non-Boussinesq, the modifications to ocean model code, advocated by MGL and illustrated by Greatbatch et al.(2001), can be implemented in conjunction with the TRM approach of MM for including the skew flux of tracers caused by the adiabatic stirring of mesoscale eddies. The paper provides the justification for simultaneously undertaking these two model improvements, and the physical interpretation of the model variables in this situation. For an example in which the Gent and McWilliams(1990) parameterization for mesoscale eddies is used in a non-Boussinesq global ocean model, consistent with the tracer equation approach advocated in Section 6 of this paper, readers are referred to Section 4b in Greatbatch et al.(2001).

2. The instantaneous equations

The treatment in MGL is based entirely on the traditional height coordinates (x, y, z), where (x, y) denotes the horizontal position and z is height above a reference geopotential surface. As in that paper, our objective is to obtain equations for averaged variables in a height-coordinate framework. However, in order to most faithfully respect the mixing characteristics of mesoscale eddies, we wish to carry out the averaging in (x, y, γ) coordinates (hereafter γ- or "isoneutral" coordinates). In γ-coordinates, the conservation equations are written with respect to the neutral tangent plane at each location. Equivalently, γ can be interpreted as the locally-referenced potential density.

It is important to realise that because our objective is to obtain a set of equations in height coordinates (i.e. at a point in space), it is only necessary for γ to be defined locally for the purposes of averaging. It follows that issues concerning the inability to define a neutral density variable globally (McDougall, 1987; McDougall and Jackett, 1998; Jackett and McDougall, 1997; Eden and Willebrand, 1999) are circumvented.

We begin by defining

$$\frac{D\gamma}{Dt} = Q, \tag{1}$$

where $\frac{D}{Dt}$ is the rate of change following a fluid particle being carried by the instantaneous flow, so that the instantaneous dianeutral velocity e (in ms^{-1}) is Q/γ_z. This definition of Q implies that if γ were a conservative variable, then $-\rho Q$ would be the divergence of the molecular flux of γ. In γ-coordinates, the instantaneous conservation equations for mass, tracer and momentum are:

$$(\frac{\rho}{\gamma_z})_t + \nabla_\gamma \cdot (\frac{\rho}{\gamma_z}\mathbf{V}) + (\frac{\rho}{\gamma_z}Q)_\gamma = 0 \tag{2}$$

$$(\frac{\rho}{\gamma_z}\tau)_t + \nabla_\gamma \cdot (\frac{\rho}{\gamma_z}\mathbf{V}\tau) + (\frac{\rho}{\gamma_z}Q\tau)_\gamma = \frac{X}{\gamma_z} \tag{3}$$

$$(\frac{\rho}{\gamma_z}\mathbf{U})_t + \nabla_\gamma \cdot (\frac{\rho}{\gamma_z}\mathbf{V}\mathbf{U}) + (\frac{\rho}{\gamma_z}Q\mathbf{U})_\gamma + 2\mathbf{\Omega} \times (\frac{\rho}{\gamma_z}\mathbf{U}) = -\frac{1}{\gamma_z}\nabla p - \mathbf{k}g\frac{\rho}{\gamma_z} + \frac{\mathbf{Y}}{\gamma_z}. \tag{4}$$

Here the notation is standard, with $\mathbf{U}$ denoting the three-dimensional, instantaneous velocity, and $\mathbf{V}$ its horizontal component (as in MM). ∇ denotes the usual three dimensional del operator and ∇_γ the two-dimensional del operator at constant γ. Since we shall later want to transform equations averaged in γ-coordinates to height coordinates, it is convenient to also write down the equivalent set of conservation equations in height coordinates; that is

$$\rho_t + \nabla \cdot (\rho\mathbf{U}) = 0 \tag{5}$$

$$(\rho\tau)_t + \nabla \cdot (\rho\mathbf{U}\tau) = X \tag{6}$$

$$(\rho\mathbf{U})_t + \nabla \cdot (\rho\mathbf{U}\mathbf{U}) + 2\mathbf{\Omega} \times (\rho\mathbf{U}) = -\nabla p - \mathbf{k}g\rho + \mathbf{Y}. \tag{7}$$

In the above, $-X$ and $-\mathbf{Y}$ are the divergences of the instantaneous molecular fluxes of τ and $\mathbf{U}$ given by $X = \nabla \cdot (\rho\kappa_\tau \nabla\tau)$ and $\mathbf{Y} = \nabla \cdot (\mu\nabla\mathbf{U}) + \frac{1}{3}\nabla(\mu\nabla\cdot\mathbf{U})$, and X could also include cross-diffusion effects such as the Soret and Dufour effects. For details on the transformation between height coordinates and a generalised coordinate system in which the depth, z, is replaced by a new variable (in our case γ), readers are referred to Appendix A of de Szoeke and Samelson(2002).

3. Mass-weighted averaging in "isoneutral" coordinates

We are now ready to average the equations in γ-coordinates. As in Section 3 of MM, we use $\tilde{\gamma}$ to denote the isoneutral surface, $\gamma = \tilde{\gamma}$, whose average height is z. In the following, $<\quad>$ is used to denote averaging at constant γ, an overbar to denote averaging at fixed height, and $\hat{}$ to denote "thickness-weighted" averaging in γ-coordinates. Noting that $\frac{1}{\gamma_z}$ is the thickness (de Szoeke and Bennett, 1993), the thickness-weighted average of any quantity, A, is defined as $\hat{A} = \tilde{\gamma}_z < A/\gamma_z >$ (see, for example, equation(28) of MM, and note that $<\gamma_z>= \tilde{\gamma}_z$).

The appearance of the factor

$$\sigma = \frac{\rho}{\gamma_z} \tag{8}$$

in equations (2)-(4) makes it natural to use σ-weighted averaging when averaging in γ-coordinates. In the context of zonal-averaging, σ-weighted averaging is referred to as "mass-weighted" averaging by Andrews et al.(1987), and is also employed by Tung(1986). Here, the averaging operator should be thought of as a long time average (Davis, 1994). (We note that our analysis is appropriate to models that do not explicitly resolve the mesoscale eddy field.) We next observe that

$$<\sigma>= \frac{\hat{\rho}}{\tilde{\gamma}_z}. \tag{9}$$

The mass-weighted averages of velocity, $\hat{\mathbf{U}}^\rho$, and tracer concentration, $\hat{\tau}^\rho$, and the deviations from these averages, are defined by

$$\hat{\mathbf{U}}^{\rho} =< \sigma \mathbf{U} > / < \sigma >, \quad \hat{\tau}^{\rho} =< \sigma \tau > / < \sigma >, \quad \mathbf{U}''' = \mathbf{U} - \hat{\mathbf{U}}^{\rho} \quad \text{and} \quad \tau''' = \tau - \hat{\tau}^{\rho}, \tag{10}$$

noting that $< \sigma \mathbf{U}''' >=< \sigma \tau''' >= 0$.

We now apply mass-weighted averaging to the instantaneous conservation equations (2)-(4) to obtain

$$(\frac{\hat{\rho}}{\tilde{\gamma}_z})_t + \nabla_{\tilde{\gamma}} \cdot (\frac{\hat{\rho}}{\tilde{\gamma}_z} \hat{\mathbf{V}}^{\rho}) + (\frac{\hat{\rho}}{\tilde{\gamma}_z} \hat{Q}^{\rho})_{\tilde{\gamma}} = 0 \tag{11}$$

$$(\frac{\hat{\rho}}{\tilde{\gamma}_z} \hat{\tau}^{\rho})_t + \nabla_{\tilde{\gamma}} \cdot (\frac{\hat{\rho}}{\tilde{\gamma}_z} \hat{\mathbf{V}}^{\rho} \hat{\tau}^{\rho}) + (\frac{\hat{\rho}}{\tilde{\gamma}_z} \hat{Q}^{\rho} \hat{\tau}^{\rho})_{\tilde{\gamma}} = -\nabla_{\tilde{\gamma}} \cdot (< \sigma \mathbf{V}''' \tau''' >) - (< \sigma Q''' \tau''' >)_{\tilde{\gamma}} \tag{12}$$

$$\begin{aligned}(\frac{\hat{\rho}}{\tilde{\gamma}_z} \hat{\mathbf{U}}^{\rho})_t + \nabla_{\tilde{\gamma}} \cdot (\frac{\hat{\rho}}{\tilde{\gamma}_z} \hat{\mathbf{V}}^{\rho} \hat{\mathbf{U}}^{\rho}) + (\frac{\hat{\rho}}{\tilde{\gamma}_z} \hat{Q}^{\rho} \hat{\mathbf{U}}^{\rho})_{\tilde{\gamma}} + 2\mathbf{\Omega} \times (\frac{\hat{\rho}}{\tilde{\gamma}_z} \hat{\mathbf{U}}^{\rho}) = -\frac{1}{\tilde{\gamma}_z} [\widehat{\nabla p}] - \mathbf{k} g \frac{\hat{\rho}}{\tilde{\gamma}_z} + \\ -\nabla_{\tilde{\gamma}} \cdot (< \sigma \mathbf{V}''' \mathbf{U}''' >) - (< \sigma Q''' \mathbf{U}''' >)_{\tilde{\gamma}}.\end{aligned} \tag{13}$$

The molecular flux terms have been dropped, but could readily be retained, for example, by an additional term, $\frac{\hat{X}}{\tilde{\gamma}_z}$ on the right hand side of (12).

4. Transforming to z-coordinates

We now transform the mass-weighted averaged equations to height coordinates. In order to do so, we note that the form of the averaged equations (11)-(13) is exactly the same as that of the instantaneous equations (2)-(4), except that instantaneous variables are replaced by mass-weighted variables, ρ is replaced by $\hat{\rho}$, and there are additional terms associated with turbulent correlations on the right hand sides. We can therefore transform directly to height coordinates using the equivalence of the instantaneous γ-coordinate equations (2)-(4) and their height-coordinate counterparts (5)-(7). However, the velocity variable in height-coordinates that results from the transformation is not $\hat{\mathbf{U}}^{\rho}$, but rather a new velocity variable $\mathbf{U^a}$ that has the same horizontal components as $\hat{\mathbf{U}}^{\rho}$, but a different vertical component, w^a. In fact

$$\mathbf{U^a} = (\hat{\mathbf{V}}^{\rho}, w^a) \tag{14}$$

where

$$w^a = \frac{\hat{D}^\rho z}{\hat{D}^\rho t}. \tag{15}$$

and

$$\frac{\hat{D}^\rho}{\hat{D}^\rho t} = \frac{\partial}{\partial t} + \hat{\mathbf{V}}^\rho \cdot \nabla_{\tilde{\gamma}} + \hat{Q}^\rho \frac{\partial}{\partial \tilde{\gamma}} \tag{16}$$

is the advective derivative following the mass-weighted average flow in γ-coordinates; that is $(\hat{\mathbf{V}}^\rho, \hat{Q}^\rho)$. It should be noted that w^a is not the same as the mass-weighted average, $\hat{w}^\rho$, of the instantaneous vertical velocity, w. Rather, w^a is the vertical velocity variable that results from the coordinate transformation. (The same issue arises in de Szoeke and Bennett(1993); compare equation in (A4) in that paper with equation (15) above).

Transforming the averaged equations to height coordinates, we now obtain:

mass conservation:

$$\hat{\rho}_t + \nabla \cdot (\hat{\rho}\mathbf{U^a}) = 0 \tag{17}$$

tracer conservation:

$$(\hat{\rho}\hat{\tau}^\rho)_t + \nabla \cdot (\hat{\rho}\mathbf{U^a}\hat{\tau}^\rho) = -\tilde{\gamma}_z \nabla_{\tilde{\gamma}} \cdot (< \sigma \mathbf{V}''' \tau''' >) - \tilde{\gamma}_z (< \sigma Q''' \tau''' >)_{\tilde{\gamma}} \tag{18}$$

momentum conservation:

$$(\hat{\rho}\hat{\mathbf{U}}^\rho)_t + \nabla \cdot (\hat{\rho}\mathbf{U^a}\hat{\mathbf{U}}^\rho) + 2\mathbf{\Omega} \times (\hat{\rho}\hat{\mathbf{U}}^\rho) = -[\widehat{\nabla p}] - \mathbf{k} g \hat{\rho} +$$
$$-\tilde{\gamma}_z \nabla_{\tilde{\gamma}} \cdot (< \sigma \mathbf{V}''' \mathbf{U}''' >) - \tilde{\gamma}_z (< \sigma Q''' \mathbf{U}''' >)_{\tilde{\gamma}} \tag{19}$$

where, in the momentum equation, the advective velocity is $\mathbf{U^a}$ (which differs from $\hat{\mathbf{U}}^\rho$ only in its vertical component). The left-hand sides of equations (17)-(18) have the same form as their density-weighted averaged counterparts in MGL (cf. equations (20)-(21) in MGL), with $\hat{\rho}$ replacing the Eulerian mean density $\overline{\rho}$, mass-weighted average tracer concentration, $\hat{\tau}^\rho$, replacing density-weighted average tracer concentration, $\overline{\tau}^\rho$, and $\mathbf{U^a}$ replacing the density-weighted averaged velocity, $\overline{\mathbf{u}}^\rho$ (see equation (19) in MGL for the definition of density-weighted averages). Likewise, the momentum conservation equation (19) is also analagous to its density-weighted counterpart (equation (22) in MGL), but is complicated by the appearance of the two velocity variables, $\mathbf{U^a}$ and $\hat{\mathbf{U}}^\rho$. On the other hand, in contrast to MGL, the right-hand side of these equations now recognise the stratified nature of oceanic motions. and very importantly, the pressure gradient in (19) is thickness-weighted (which leads to the form stress term in equation (28)).

At this stage, it is convenient to recognise that since our averaged equations apply to the large scale flow of the ocean, we can simplify (19) by (i) making the hydrostatic approximation and (ii) neglecting the contribution from the mass-weighted average vertical velocity, $\hat{w}^\rho$, to the Coriolis term (Gill, 1982, Chapter 11). Since the two velocity variables, $\mathbf{U^a}$ and $\hat{\mathbf{U}}^\rho$, differ only in their vertical components, making these simplifications allows us to work with the single velocity variable, $\mathbf{U^a} = (\hat{\mathbf{V}}^\rho, w^a)$. We therefore replace (19) by

$$(\hat{\rho}\hat{\mathbf{V}}^\rho)_t + \nabla \cdot (\hat{\rho}\mathbf{U^a}\hat{\mathbf{V}}^\rho) + 2\mathbf{\Omega} \times (\hat{\rho}\hat{\mathbf{V}}^\rho) = -[\widehat{\nabla_H p}]$$
$$-\tilde{\gamma}_z \nabla_{\tilde{\gamma}} \cdot (< \sigma \mathbf{V}''' \mathbf{V}''' >) - \tilde{\gamma}_z (< \sigma Q''' \mathbf{V}''' >)_{\tilde{\gamma}}, \tag{20}$$

and the hydrostatic equation

$$0 = -\widehat{p_z} - g\hat{\rho}, \tag{21}$$

where ∇_H is the horizontal gradient operator. Further discussion of the fully non-hydrostatic equations can be found in Section 6.

We now follow the same procedure as in MGL and define a new velocity variable

$$\mathbf{U}^* = \frac{\hat{\rho}\mathbf{U^a}}{\rho_o}. \tag{22}$$

$\mathbf{U}^*$ has horizontal and vertical components $\mathbf{V}^* = \frac{\hat{\rho}\hat{\mathbf{V}}^\rho}{\rho_o}$ and $w^* = \frac{\hat{\rho} w^a}{\rho_o}$, respectively, and is analogous to $\tilde{\overline{\mathbf{u}}}$ defined by (23) in MGL. ρ_o is a constant reference density. Using this velocity variable, (17), (18) and (20) become

$$(\frac{\hat{\rho}}{\rho_o})_t + \nabla \cdot \mathbf{U}^* = 0 \tag{23}$$

$$(\frac{\hat{\rho}}{\rho_o}\hat{\tau}^\rho)_t + \nabla \cdot (\mathbf{U}^*\hat{\tau}^\rho) = -\frac{\tilde{\gamma}_z}{\rho_o}\nabla_{\tilde{\gamma}} \cdot (< \sigma \mathbf{V}'''\tau''' >) - \frac{\tilde{\gamma}_z}{\rho_o}(< \sigma Q'''\tau''' >)_{\tilde{\gamma}} \tag{24}$$

and

$$\begin{aligned} \mathbf{V}^*{}_t + \nabla \cdot (\frac{\rho_o}{\hat{\rho}}\mathbf{U}^*\mathbf{V}^*) + 2\mathbf{\Omega} \times \mathbf{V}^* = -\frac{1}{\rho_o}[\widehat{\nabla_H p}] - \frac{\tilde{\gamma}_z}{\rho_o}\nabla_{\tilde{\gamma}} \cdot (< \sigma \mathbf{V}'''\mathbf{V}''' >) \\ -\frac{\tilde{\gamma}_z}{\rho_o}(< \sigma Q'''\mathbf{V}''' >)_{\tilde{\gamma}}. \end{aligned} \tag{25}$$

So far, we have left the turbulent correlation terms in their γ-coordinate formulation. We now follow MM and parameterize these terms using symmetric diffusion tensors, $\mathbf{S}^\tau$ for tracer and $\mathbf{S}^\mathbf{u}$ for momentum. We also note, following MM, that the thickness-weighted horizontal pressure gradient $[\widehat{\nabla_H p}]$ in (25) can be written as the sum of the Eulerian mean horizontal pressure gradient $\nabla_H \overline{p}$ plus a form stress term. Likewise, we follow MM and replace the thickness-weighted vertical pressure gradient in the hydrostatic equation (21) by the Eulerian mean vertical pressure gradient (there is further discussion on this point in Section 6, following equation (50)). Equations(23)-(25) and (21) can now be written as

$$(\frac{\hat{\rho}}{\rho_o})_t + \nabla \cdot \mathbf{U}^* = 0 \tag{26}$$

$$(\frac{\hat{\rho}}{\rho_o}\hat{\tau}^\rho)_t + \nabla \cdot (\mathbf{U}^*\hat{\tau}^\rho) = \nabla \cdot (\mathbf{S}^\tau \nabla \hat{\tau}^\rho) \tag{27}$$

$$\mathbf{V}^*{}_t + \nabla \cdot (\frac{\rho_o}{\hat{\rho}}\mathbf{U}^*\mathbf{V}^*) + 2\mathbf{\Omega} \times \mathbf{V}^* = -\frac{1}{\rho_o}\nabla_H \overline{p} + \nabla \cdot (\mathbf{S}^\mathbf{u}\nabla \frac{\rho_o}{\hat{\rho}}\mathbf{V}^* - \mathbf{k}\ \mathbf{F}) \tag{28}$$

$$0 = -\overline{p}_z - g\hat{\rho}. \tag{29}$$

Here $\mathbf{k}$ is a unit vector in the upwards vertical direction, and $\mathbf{F}$ is a two dimensional vector in the horizontal plane representing the eddy form stress. As in MM, the diffusion tensors $\mathbf{S}^\tau$ and $\mathbf{S}^\mathbf{u}$ are the symmetric diffusion tensors of Redi(1982), combining diffusion along the local γ-surface (or local neutral tangent plane) with very much weaker isotropic diffusion by small-scale turbulence which is commonly referred to as diapycnal diffusion. The factor $\frac{\rho_o}{\hat{\rho}}$ in the $\mathbf{S^u}$ term ensures that kinetic energy is dissipated in the ocean interior (see Greatbatch et al.(2001), Section 3c, and note that kinetic energy for this system of equations is given by $KE = \frac{1}{2}\hat{\rho}\ \hat{\mathbf{V}}^\rho \cdot \hat{\mathbf{V}}^\rho$).

As in MGL, we note that in steady state, equations (26)-(29) differ from those currently carried by hydrostatic Boussinesq ocean models that include form drag only by the factor of $\frac{\rho_o}{\hat{\rho}}$ in the momentum flux divergence terms, even though these equations are, in fact, fully non-Boussinesq. We also note, following MGL, that the Boussinesq approximation corresponds to replacing $\frac{\hat{\rho}}{\rho_o}$ by 1 everywhere in (26)-(29) except in the gravitational acceleration term, and that there is no systematic error associated with making the Boussinesq approximation in unsteady situations (see Section 6 of MGL and note that the analysis in MGL carries over with a suitable reinterpretation of the variables). An additional feature is the eddy form stress term $-\nabla \cdot (\mathbf{k}\ \mathbf{F})$ associated with the eddies. The appearance of this term is not a consequence of the use of non-Boussinesq equations of motion. In fact, it also appears in MM, equation (66), where $\mathbf{F} = -f\mathbf{k} \times \mathbf{\Psi}$, $\mathbf{\Psi}$ is the quasi-Stokes streamfunction (see Section 5) and f is the Coriolis parameter. The duality between applying an eddy parameterization in either the tracer or momentum equations has been noted by Greatbatch and Lamb(1990), Gent et al.(1995), Greatbatch(1998) and Greatbatch(2001) (the latter in the context of averaging at fixed height). The traditional approach is to add the advection associated with the eddy-induced transport to the tracer equation (e.g. Danabasoglu and McWilliams, 1995; Hirst and McDougall, 1996), where it can also be written as a skew diffusion (e.g. Griffies, 1998; MM), an issue we explore in Section 6. In the case of the TRM, putting the eddy parameterization in the momentum equation arises quite naturally, and has the advantage, as in MM that, at least in the hydrostatic case, the averaged equations (26)-(29) use only one velocity variable, namely $\mathbf{U}^*$. This contrasts with the usual situation, where writing the averaged momentum equation in a form to take account of the eddy forcing requires carrying two velocity variables, as in the Transformed Eulerian Mean (Andrews et al., 1987), and as in Greatbatch(2001).

5. Physical Interpretation and Boundary Conditions

We begin by noting that the horizontal component of the velocity variable $\mathbf{U}^*$, can be written

$$\mathbf{V}^* = \frac{\hat{\rho}\hat{\mathbf{V}}^{\rho}}{\rho_o} = \tilde{\gamma}_z < \frac{1}{\gamma_z}\frac{\rho\mathbf{V}}{\rho_o} > . \tag{30}$$

In other words, $\mathbf{V}^*$ is the thickness-weighted average on a γ-surface of $\frac{\rho}{\rho_o}\mathbf{V}$. We now apply the analysis in Section 2 of MM to the horizontal velocity variable $\frac{\rho}{\rho_o}\mathbf{V}$, rather than $\mathbf{V}$, as done there. We therefore consider the surface $\gamma = \gamma_a$ whose average height is z_a and whose instantaneous height is $z_a + z_a'$ ($\overline{z_a'} = 0$). Taking the ocean bottom to be at $z = -H$, we follow MM to obtain

$$\overline{\int_{-H}^{z_a+z_a'} \frac{\rho}{\rho_o}\mathbf{V}dz} = \int_{-H}^{z_a} \overline{\frac{\rho}{\rho_o}\mathbf{V}}dz + \overline{\int_{z_a}^{z_a+z_a'} \frac{\rho}{\rho_o}\mathbf{V}dz} \tag{31}$$

where overbar denotes the usual long time average. The term on the left hand side is the scaled (by a factor ρ_o) average horizontal mass transport per unit area below the surface $\gamma = \gamma_a$. As in MM, we now define a quasi-Stokes streamfunction $\mathbf{\Psi}$ by

$$\mathbf{\Psi}(z_a) = \overline{\int_{z_a}^{z_a+z_a'} \frac{\rho}{\rho_o}\mathbf{V}dz} \tag{32}$$

so that

$$\overline{\int_{-H}^{z_a+z_a'} \frac{\rho}{\rho_o}\mathbf{V}dz} = \int_{-H}^{z_a} \overline{\frac{\rho}{\rho_o}\mathbf{V}}dz + \mathbf{\Psi}(z_a) \tag{33}$$

$\mathbf{\Psi}(z_a)$ is the (scaled) horizontal mass transport below the $\gamma = \gamma_a$ surface that is associated with the eddies.

Exactly as in MM, we can go further and consider the average horizontal mass transport between isopycnals, that is

$$\overline{\int_{z_b+z_b'}^{z_a+z_a'} \frac{\rho}{\rho_o}\mathbf{V}dz} = \int_{z_b}^{z_a} \overline{\frac{\rho}{\rho_o}\mathbf{V}}dz + \mathbf{\Psi}(z_a) - \mathbf{\Psi}(z_b). \tag{34}$$

The left hand side of this equation is the thickness-weighted (scaled) horizontal mass flux between the surfaces $\gamma = \gamma_a$ and $\gamma = \gamma_b$, so that dividing by $(z_a - z_b)$ and using (30) we obtain in the limit $(z_a - z_b) \to 0$

$$\mathbf{V}^* = \frac{\hat{\rho}\hat{\mathbf{V}}^\rho}{\rho_o} = \overline{\tilde{\mathbf{V}}} + \mathbf{\Psi}_z. \tag{35}$$

Here $\overline{\tilde{\mathbf{V}}}$ is the horizontal component of the averaged velocity variable

$$\overline{\tilde{\mathbf{U}}} = \frac{\overline{\rho \mathbf{U}}}{\rho_o} \tag{36}$$

introduced by equation (23) of MGL, and which MGL argue is the velocity variable that is carried by models (note that here $\tilde{}$ is used to be consistent with MGL, not to denote isoneutral averaging. In fact, $\overline{\tilde{\mathbf{U}}}$ is averaged at fixed height). By analogy with MM, it follows that the height-averaged velocity $\overline{\tilde{\mathbf{U}}}$ corresponds to the Eulerian mean velocity $\overline{\mathbf{U}}$ in MM. We therefore define the quasi-Stokes velocity by

$$\mathbf{U}^{\mathrm{s}} = \mathbf{U}^* - \overline{\tilde{\mathbf{U}}}. \tag{37}$$

The quasi-Stokes velocity corresponds to what is usually called the eddy-induced transport velocity, and is the velocity that must be added to $\overline{\tilde{\mathbf{U}}}$ to take account of the eddy-induced transport. Clearly the horizontal component of $\mathbf{U}^{\mathrm{s}}$ (i.e. $\mathbf{V}^{\mathrm{s}}$) is given by

$$\mathbf{V}^{\mathrm{s}} = \mathbf{\Psi}_z. \tag{38}$$

We next note that, exactly as in MM, an approximate expression for the quasi-Stokes streamfunction, valid to cubic order in perturbation amplitude α, is

$$\mathbf{\Psi} = \frac{1}{\rho_o}\left[-\frac{\overline{(\rho\mathbf{V})'\gamma'}}{\overline{\gamma}_z} + \frac{(\overline{\rho\mathbf{V}})_z}{\overline{\gamma}_z}\left(\frac{\overline{\phi}}{\overline{\gamma}_z}\right)\right] + O(\alpha^3) \tag{39}$$

where overbar denotes average at fixed height, $'$ denotes perturbation at fixed height, and $\phi = \frac{1}{2}\overline{\gamma'^2}$ is half the variance of γ at height z.

As in MM, the physical interpretation provides guidance as to how the quasi-Stokes streamfunction behaves as the surface and bottom boundaries are approached (see

Section 8 of MM). As in MM, we recommend that the quasi-Stokes streamfunction be tapered to zero on these boundaries, and that the tapering take place over the boundary layer defined by the average height of γ-surfaces that outcrop along these boundaries during the averaging period used to define $\mathbf{U}^*$. This has consequences for our averaged momentum equation, i.e. (28). As shown in MM, assuming the eddies to be in geostrophic balance enables the form stress, $\mathbf{F}$, arising from the thickness-weighted average of the horizontal pressure gradient, to be expressed in terms of the quasi-Stokes streamfunction as

$$\mathbf{F} = -f\mathbf{k} \times \mathbf{\Psi}. \tag{40}$$

The argument carries over directly to the non-Boussinesq case, leading to exactly the same expression, i.e. (40) (noting that geostrophic balance for the instantaneous flow takes the form $f\mathbf{k} \times \rho\mathbf{V} = -\nabla_H p$ in the non-Boussinesq system (see equation (7)). It should also be noted that a more general expression for $\mathbf{F}$ in terms of $\mathbf{\Psi}$ can be obtained by assuming a balance between the Coriolis, horizontal pressure gradient and a linear (Rayleigh) friction term, i.e. $f\mathbf{k} \times \rho\mathbf{V} = -\nabla_H p - \epsilon\rho\mathbf{V}$. With $\epsilon \neq 0$, $\mathbf{F}$ no longer goes to zero on the equator (where $f = 0$), and the angle between the direction of $\mathbf{F}$ and that of $\mathbf{\Psi}$ now depends on the ratio of f to the Rayleigh friction coefficient, ϵ.

Finally, we note, as in Appendix B of MM, that since the tracer variables are mass-weighted averages, it is natural to interpret the density computed using the equation of state carried by a model, i.e. $\rho = \rho(\hat{S}^\rho, \hat{\theta}^\rho, \overline{p})$, as the thickness-weighted average density $\hat{\rho}$ (the correspondence is not exact, but, as argued in MM, the difference is not likely to be important in practice). In hydrostatic models, this is the density that would be used in the computation of the horizontal pressure gradient terms carried by the model.

The bottom and free surface kinematic boundary conditions follow (with a suitable reinterpretation of the variables) as in Greatbatch, Lu and Cai(2001). At the bottom,

$z = -H$, we have

$$w^* = -\mathbf{V}^* \cdot \nabla_H H \qquad at \quad z = -H \tag{41}$$

(equivalently, using (16), $\hat{D}^\rho(z+H)/\hat{D}^\rho t = 0$). At the surface we need to define what we mean by the sea surface height. We do this, following Lu(2001) and Greatbatch, Lu and Cai(2001), by choosing the sea surface height variable, here denoted η^a, so that

$$\frac{\partial}{\partial t}\int_{-H}^{\eta^a} \frac{\hat{\rho}}{\rho_o} dz + \nabla_H \cdot \int_{-H}^{\eta^a} \mathbf{V}^* dz = -(\overline{E} - \overline{P} - \overline{R})/\rho_o \tag{42}$$

where E, P and R denote the mass flux per unit area associated with evaporation, precipitation and river run-off, respectively. Combining (41) and (42) with the vertical integral of the continuity equation (26) then gives

$$w^* = (\hat{\rho}/\rho_o)\eta^a_t + \mathbf{V}^* \cdot \nabla_H \eta^a + (\overline{E} - \overline{P} - \overline{R})/\rho_o \qquad at \quad z = \eta^a. \tag{43}$$

We noted in passing that the sea surface height variable, η^a, is a different sea surface height variable from that in Greatbatch, Lu and Cai(2001), because of the different averaging used for the variables that appear in equation (42).

6. Formulation based on the tracer equation

As noted in Section 4, the traditional approach to including the effects of mesoscale stirring in coarse-resolution ocean models is to add the advection associated with the eddy-induced transport to the tracer equation (e.g. Danabasoglu and McWilliams, 1995; Hirst and McDougall, 1996), rather than to parameterize the eddy forcing in the averaged momentum equation, as in (26)-(29). In a Boussinesq framework, the eddy-induced advection in the tracer equation can be written as a skew diffusion, as discussed by Griffies(1998) and MM. Here, we outline the tracer equation approach in the context of the non-Boussinesq formalism developed in this paper.

We begin by using (37) to write the averaged mass and tracer conservation equations (26) and (27) as

$$(\frac{\hat{\rho}}{\rho_o})_t + \nabla\cdot(\overline{\tilde{\mathbf{U}}} + \mathbf{U}^s) = 0 \tag{44}$$

and

$$(\frac{\hat{\rho}}{\rho_o}\hat{\tau}^\rho)_t + \nabla\cdot[(\overline{\tilde{\mathbf{U}}} + \mathbf{U}^s)\hat{\tau}^\rho] = \nabla\cdot(\mathbf{S}^\tau\nabla\hat{\tau}^\rho) \tag{45}$$

where $\mathbf{U}^s$ is the quasi-Stokes velocity. In addition, we require the averaged momentum equation, corresponding to the Eulerian mean momentum equation in height coordinates. From equation(26) of MGL, this is

$$\overline{\tilde{\mathbf{U}}}_t + \nabla\cdot(\frac{\rho_o}{\overline{\rho}}\overline{\tilde{\mathbf{U}}}\;\overline{\tilde{\mathbf{U}}}) + 2\mathbf{\Omega}\times\overline{\tilde{\mathbf{U}}} = -\frac{1}{\rho_o}\nabla\overline{p} - \mathbf{k}g\frac{\overline{\rho}}{\rho_o} + \nabla\cdot(\mathbf{A}\nabla\frac{\rho_o}{\overline{\rho}}\overline{\tilde{\mathbf{U}}}). \tag{46}$$

We also need the continuity equation satisfied by $\overline{\tilde{\mathbf{U}}}$; that is equation (24) in MGL, namely

$$(\frac{\overline{\rho}}{\rho_o})_t + \nabla\cdot\overline{\tilde{\mathbf{U}}} = 0. \tag{47}$$

An immediate complication is the appearance of the Eulerian mean density $\overline{\rho}$ in (46) and (47), rather than $\hat{\rho}$, as in (44) and (45). One consequence is that, in general, the quasi-Stokes velocity $\mathbf{U}^s$ is divergent and satisfies

$$\nabla\cdot\mathbf{U}^s = -\frac{\partial}{\partial t}\left(\frac{\hat{\rho}-\overline{\rho}}{\rho_o}\right), \tag{48}$$

as can easily be verified from (44) and (47).

Since the density variable available to the model is identified as $\hat{\rho}$, rather than $\overline{\rho}$, it follows immediately that a parameterization for $\overline{\rho}$ is required. The simplest such parameterization is to put $\overline{\rho} = \hat{\rho}$. Doing so incurs an error that usually is quadratic in perturbation amplitude in the ocean interior (see equation (28) in MM), but which increases from second to first order as the surface and bottom boundaries are approached (Killworth, 2001). The boundary layers over which the error increases are identical to the boundary layers over which the quasi-Stokes streamfunction, $\mathbf{\Psi}$ should be tapered to zero (i.e. the boundary layers discussed in Section 8 of MM that arise from the transient outcropping of γ-surfaces associated with the eddies). To compare $\overline{\rho}$ and $\hat{\rho}$, we can use Figure 3 in Killworth's paper. The figure shows a comparison between $\overline{\rho}$ and $\hat{\rho}$ (with temperature substituted for ρ) from an eddy-resolving channel model experiment. As noted by Killworth, over most of the domain, $\overline{\rho}$ and $\hat{\rho}$ are barely distinguishable from each other, so that replacing $\overline{\rho}$ by $\hat{\rho}$ is almost certainly an acceptable approximation there. It is only near the surface that $\overline{\rho}$ and $\hat{\rho}$ differ significantly, differences reaching as much as 1 $kg\ m^{-3}$. Since both $\overline{\rho}$ and $\hat{\rho}$ are of order $10^3\ kg\ m^{-3}$, the implication is that the error incurred in replacing $\overline{\rho}$ by $\hat{\rho}$ is expected to be about 0.1%. We can therefore replace the factor $\overline{\rho}$ by $\hat{\rho}$ in the two momentum flux divergence terms in (46) without incurring significant error (certainly, the error does not compromise the fact we are working in a non-Boussinesq regime, since the Boussinesq approximation leads to an error that is about a factor of 30 larger). An extension of the argument in Appendix B of MM also allows us to replace $\overline{\rho}$ by $\hat{\rho}$ in the gravitational acceleration term (but see the later discussion of this point after (50)). We now wish to replace $\overline{\rho}_t$ by $\hat{\rho}_t$ in (47). In a statistically steady state ($\frac{\partial}{\partial t} = 0$), there is clearly no error in doing this, since both terms are then zero. In unsteady situations, the same argument as above shows that the error is significantly less than the error in making the Boussinesq approximation, and is, therefore, acceptable. Likewise, the analysis in Section 6 of MGL, adapting the method of Lu(2001), can be used to show that there is no systematic error in the diapycnal transport of tracer or momentum involved in this replacement.

In light of the above discussion, and in addition making the hydrostatic approximation, we therefore replace (46) by

$$\overline{\overline{\mathbf{V}}}_t + \nabla\cdot(\frac{\rho_o}{\hat{\rho}}\overline{\overline{\mathbf{U}}}\;\overline{\overline{\mathbf{V}}}) + 2\mathbf{\Omega}\times\overline{\overline{\mathbf{V}}} = -\frac{1}{\rho_o}\nabla_H\overline{p} + \nabla\cdot(\mathbf{A}\nabla\frac{\rho_o}{\hat{\rho}}\overline{\overline{\mathbf{V}}});$$
$$0 = -\overline{p}_z - g\hat{\rho} \qquad (49)$$

and note that, to the same level of approximation, $\nabla\cdot\mathbf{U}^{\mathbf{s}} = 0$, so that

$$\mathbf{V}^{\mathbf{s}} = \mathbf{\Psi}_z; \qquad w^s = -\nabla_H\cdot\mathbf{\Psi}. \qquad (50)$$

It follows that parameterization of $\mathbf{U^s}$ (required in (44) and (45)) amounts to parameterizing the quasi-Stokes streamfunction. Also, since the quasi-Stokes velocity is non-divergent at this level of approximation, the contribution of $\mathbf{U^s}$ to the mass conservation equation, (44), can be ignored, while its contribution to the tracer equation, (45), can be written in terms of a skew-diffusivity, as in Griffies (1998) and MM.

In going from (46) to (49), we have replaced $\overline{\rho}$ by $\hat{\rho}$ in the gravitational acceleration term. An argument to justify this can be found in Appendix B of MM. MM's argument applies in the ocean interior, where $\overline{\rho}$ and $\hat{\rho}$ do not differ greatly. The same argument can be extended to the surface, provided the depth over which $\overline{\rho}$ and $\hat{\rho}$ differ significantly is not too deep (scale depth of order 10's of meters), and the horizontal scale over which the difference between $\overline{\rho}$ and $\hat{\rho}$ varies is not too small (on the order of 1000 kms). The same argument can be used to justify going from equation (21) to equation (29) in Section 4. (As MM point out, the error being discussed here actually appears as an error in the horizontal pressure gradient term and so averages to zero when an area average is taken over an area of high eddy activity.) It should be noted that although we made the hydrostatic approximation in writing down (49), nothing in our discussion of replacing $\overline{\rho}$ by $\hat{\rho}$ depends on making the hydrostatic approximation. In this sense, the tracer equation approach discussed here is more suitable to nonhydrostatic regimes than the approach taken in Section 4 where the hydrostatic approximation was used to eliminate the need to carry two different velocity variables in (19).

The kinematic condition at the surface is obtained from (37) and (43) and is

$$\overline{\tilde{w}} + w^s = (\hat{\rho}/\rho_o)\eta^a_t + (\overline{\tilde{\mathbf{V}}} + \mathbf{V^s}) \cdot \nabla_H \eta^a + (\overline{E} - \overline{P} - \overline{R})/\rho_o \qquad at \quad z = \eta^a. \tag{51}$$

Since the quasi-Stokes streamfunction is tapered to zero on the boundaries, we expect $w^s = \mathbf{V^s} \cdot \nabla_H \eta^a$ at the surface, so that

$$\overline{\tilde{w}} = (\hat{\rho}/\rho_o)\eta^a_t + \overline{\tilde{\mathbf{V}}} \cdot \nabla_H \eta^a + (\overline{E} - \overline{P} - \overline{R})/\rho_o \qquad at \quad z = \eta^a. \tag{52}$$

The kinematic boundary condition at the bottom is the same as equation (14) in

Greatbatch, Lu and Cai(2001), i.e.

$$\overline{\overline{w}} = -\overline{\tilde{\mathbf{V}}} \cdot \nabla_H H \quad at \quad z = -H. \tag{53}$$

The latter is obtained by applying density-weighted averaging at fixed height to the bottom kinematic condition applying to the instantaneous flow.

7. Summary and discussion

We have shown how the Temporal-Residual-Mean of McDougall and McIntosh(2001) can be combined with the methodology of McDougall, Greatbatch and Lu(2002) so that the non-Boussinesq equations of motion can be averaged in a way that respects the fact that mesoscale eddies stir almost exclusively along the neutral tangent plane. Since our averaged equations are written in height coordinates, we need only define neutral surfaces locally, for the purposes of averaging, avoiding the difficulty that neutral surfaces cannot, in general, be defined globally (McDougall, 1987; McDougall and Jackett, 1988; Jackett and McDougall, 1997; Eden and Willebrand, 1999). We note that the tracers carried by a model should be interpreted as the mass-weighted, isoneutral-averaged tracer, and the horizontal velocity as the thickness-weighted average of the horizontal mass flux per unit area, scaled by a representative density for sea water. The horizontal component of the quasi-Stokes velocity is the difference between this velocity and the height average of the new velocity variable introduced by MGL, namely the horizontal mass flux per unit area averaged at constant height, scaled by the same reference density. The quasi-Stokes velocity represents the eddy-induced transport by the eddies, and its horizontal component can be expressed in terms of a quasi-Stokes streamfunction that has an exact correspondence with the quasi-Stokes streamfunction introduced by McDougall and McIntosh(2001). In view of this correspondence, the same argument regarding the behaviour of the quasi-Stokes streamfunction near the surface and bottom boundaries applies here, as it does in McDougall and McIntosh(2001), and is one of the advantages of the present approach.

It should be noted that the treatment given in McDougall, Greatbatch and Lu(2002) of the averaged non-Boussinesq equations of motion, and the analysis of the errors associated with making the Boussinesq approximation given in that paper, carry over to the present paper without change. The only difference is the presence of the form drag term in (28) and the different interpretation of the averaged variables: $\hat{\rho}$, $\mathbf{U}^*$ and $\hat{\tau}^\rho$ here, compared with $\overline{\rho}$, $\overline{\tilde{\mathbf{U}}}$ and $\overline{\tau}^\rho$ there. The reason the analysis carries over without change is the correspondence between the averaged equations (26)-(29) here, and (20)-(22) in McDougall, Greatbatch and Lu(2002). Likewise, the method of Greatbatch, Lu and Cai(2001) can be used to easily modify existing code for a Boussinesq, hydrostatic ocean model to make it non-Boussinesq and consistent with the averaging being proposed here. In Greatbatch, Lu and Cai(2001), the modifications are applied to the POP code (see http://www.acl.lanl.gov/climate/models/pop/). The modifications have also been applied in the MOM4 code where the cpu overhead is only a few per cent (Griffies, personal communication; see http://www.gfdl.noaa.gov/MOM/MOM.html).

An interesting aspect of our study is the finding that the approach to mesoscale eddy parameterization based on the averaged momentum equation (e.g. Greatbatch and Lamb, 1990) is more straightforward than that based on the averaged tracer equation. The tracer equation approach is complicated by the appearance of two different density variables, $\hat{\rho}$ and $\overline{\rho}$, the second of which is not available to the model. Nevertheless, we argued that the error incurred in replacing $\overline{\rho}$ by $\hat{\rho}$ is small compared to the error associated with the Boussinesq approximation, and is, therefore, acceptable. In the momentum equation approach, the non-hydrostatic case is complicated by the appearance of two different vertical velocity variables in the averaged equations (17)-(19). This particular complication is removed when the hydrostatic approximation is applied to the averaged equations and the contribution from the vertical velocity is neglected in the Coriolis term. Both these simplifications apply to the large-scale flow of the ocean, of interest here. Issues we have not addressed include exactly how the quasi-Stokes streamfunction should be tapered to zero at the surface and bottom boundaries, and how surface fluxes can be parameterized in terms of the model variables. These issues are intimately connected with the difficult problem if deciding how the outcropping layers at the surface and bottom boundaries should be treated, and are left for a future study.

Recently, de Szoeke and Samelson(2002) have pointed out that when the hydrostatic approximation is made, there is a duality between the Boussinesq and non-Boussinesq equations of motion. The duality is easily seen when the hydrostatic, non-Boussinesq equations of motion are written using pressure, p, as the vertical coordinate. In particular, the continuity equation then appears (in plane Cartesian coordinates) as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{54}$$

where $\omega = \frac{Dp}{Dt}$, and so has the same form as the Boussinesq continuity equation written in height-coordinates (see Holton(1981) for a detailed discussion of pressure coordinates as used in meteorology). Assuming that the instantaneous flow is hydrostatic, and taking the instantaneous non-Boussinesq equations of motion written in pressure coordinates as the starting point, it is straightforward to apply the machinery of MM to obtain an alternative formulation of the TRM appropriate to a non-Boussinesq ocean, with the pressure, p, replacing the height, z, throughout MM's analysis, and isobaric surfaces replacing geopotential surfaces throughout the interpretation. Since, from the hydrostatic approximation, $dp = -g\rho dz$, the factors of ρ that appear in our analysis, but not in the Boussinesq analysis of MM, would also appear in the pressure coordinate analysis, following conversion to z-coordinates (note, in particular, that the thickness in pressure coordinates is $-\frac{1}{\gamma_p} = g\rho\frac{1}{\gamma_z}$, which differs from equation (8) only by the scaling factor g). Indeed, the thickness-weighted, isoneutral-averaged variables in the pressure coordinate case are identical to the mass-weighted, isoneutral-averaged variables in this paper. On the other hand, the interpretation of the Eulerian mean velocity, and hence of the quasi-Stokes streamfunction, differs between the two cases. In our paper, the horizontal component of the Eulerian mean velocity is the scaled horizontal mass flux per unit area averaged at fixed height, whereas in the pressure coordinate approach, the horizontal component of the Eulerian mean velocity is the average at constant pressure of the horizontal velocity. The difficulty, discussed in Section 4, of having two different density variables applies in the pressure coordinate case, as in the z-coordinate approach. A disadvantage of the pressure-coordinate approach is the need for the instantaneous flow to be hydrostatic.

Finally, we note that the Boussinesq TRM formulation of McDougall and McIntosh(2001) is recovered by replacing ρ by ρ_o everywhere in this paper, except in the gravitational acceleration term.

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