On Conservation Equations in Oceanography: - How Accurate are Boussinesq Ocean Models?

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Abstract.

Traditionally, the conservation equations in oceanography include the Boussinesq approximation, and the velocity variable is interpreted as the Eulerian mean velocity averaged over turbulent scales. If such a view is adopted, then the conservation equations for tracers contain errors that are often as large as the diapycnal mixing term. This result has been known for about a decade and, at face value, implies that all Boussinesq ocean models contain leading order errors in their conservation equations. To date there has not yet been a solution proposed to avoid this conundrum. Here it is shown that the conundrum can be solved by interpreting the horizontal velocity vector carried by Boussinesq ocean models as the average horizontal mass flux per unit area divided by the constant reference density that appears in the horizontal momentum equation. We argue that the vector labelled the "velocity" in present ocean models is not, and never was, the Eulerian mean velocity. If it were, then the conservation equations for salinity anomaly and potential temperature would contain systematic errors whose magnitude would be as large as the diapycnal mixing terms. By interpreting the model's horizontal "velocity" as being proportional to the horizontal mass flux per unit area, the conservation equations in the present generation of Boussinesq models are actually much more accurate even than previously thought. In particular, when these Boussinesq models achieve a steady state, they are actually almost fully non-Boussinesq, and in a non-steady state there is no systematic error in the diapycnal advective/diffusive balance due to the Boussinesq approximation. With the above interpretation of the model's "velocity", it is also relatively simple to change the model code to make it fully non-Boussinesq even when the flow is unsteady.

A conclusion of our work is that the Boussinesq approximation actually consists of three parts, not two, as has been assumed in the past. Traditionally, the Boussinesq approximation consists of replacing (i) the equation for conservation of mass by the equation for conservation of volume and (ii) the density that appears in the temporal

and advection operators by a constant reference density. Here we show that it is also important to (iii) ensure that using a divergence free velocity to advect tracer does not lead to significant error, an aspect of the Boussinesq approximation that has previously been overlooked.

1. Introduction

The conservation equations of oceanography are treated in many texts, including Batchelor (1967) and Gill (1982). In theoretical analysis and numerical modelling, it is traditional to make the Boussinesq approximation where every appearance of density is replaced by a fixed reference density except in the buoyant force in the vertical momentum equation. Since the reference density is usually taken to be 1000 kg m^{-3} and since the in-situ density of seawater can be as large as 1050 kg m^{-3}, it appears as though the Boussinesq approximation leads to errors in the conservation equations which are typically 3% but can be as large as 5%.

Spiegel and Veronis (1960) examined the conditions under which the effect of fluid compressibility can be neglected in the mass conservation equation, the equation for potential temperature, and the inertia term in the momentum equations. However, their work did not consider the averaged equations in a turbulent fluid, and, more seriously, they did not examine the error in the potential temperature equation that arises from using a divergence free velocity as the advecting velocity. Concern about neglecting the velocity divergence in the averaged Boussinesq tracer equation has been raised by McDougall and Garrett (1992; hereafter MG) and Davis (1994). Using scale analysis, MG found that the flux form of the averaged Boussinesq tracer equation is in error by at least 30% in comparison with the diapycnal mixing term due to the neglect of the divergence of the Eulerian mean velocity. MG were rather horrified to imagine that the conservation equations that we use in oceanography could be in error by 30% due to the Boussinesq approximation and they advanced the following (false) argument which has since been debunked by Davis (1994, see his section 3e). MG noted that the advective form of the scalar conservation equations does not contain a term involving the divergence of the Eulerian mean velocity, $\nabla \cdot \overline{\mathbf{u}}$, and suggested that so long as $\nabla \cdot \overline{\mathbf{u}} = 0$ is rigorously enforced, then the erroneous divergence forms of the conservation statements would reduce to the correct advective forms and so all would be well. In short, MG were saying that two wrongs (the false assertion that $\nabla \cdot \overline{\mathbf{u}} = 0$ in two separate places) would cancel each other. Davis (1994) pointed out that if accurate observations of the horizontal Eulerian-mean velocity were available everywhere, and if the flux form of the averaged Boussinesq tracer equation were vertically integrated over a region bounded by an isoline of the tracer in order to deduce the diapycnal property flux, then the missing term would lead to a mis-estimation of the diapycnal property flux that is as large as that expected with canonical values of the diapycnal diffusion coefficient. This argument of Davis (1994) is the same point that is made above, namely that the term involving $\nabla \cdot \overline{\mathbf{u}}$ is of the same order (MG say at least 30%) as the diapycnal mixing term in the averaged conservation equations. The argument of Davis (1994) is correct as it stands and MG must stand corrected:- the two wrongs that they identified do not make a right.

In this note, we reexamine the important issue of the averaged conservation equations as they apply to the ocean, and, in particular, the accuracy of the equations solved by currently existing ocean models, which commonly include the Boussinesq approximation. Following a review of the Boussinesq approximation in Section 2, we revisit the issue raised by MG in Section 3, but this time concentrate on the effect of using a divergence-free velocity, such as carried by Boussinesq models, to advect the averaged tracer field. If, as is traditional, one assumes that the horizontal momentum equations of an ocean model are prognostic equations for the Eulerian-mean (that is Reynolds averaged) horizontal velocity, the method of Lu (2001) can be used to confirm that the model's tracer conservation equation is indeed in error by an amount equivalent to the magnitude of the diapycnal mixing term, but now, in contrast to MG, the error is independent of any constant offset of the tracer. We note that the relative magnitude of the error is enhanced by the large cancellation between the horizontal and vertical advection terms in the model's tracer equation on sloping isopycnal surfaces, leaving a much smaller advection (of Boussinesq magnitude with the conventional interpretation of the model's variables) to balance the diapycnal mixing. We then go further in Section

4 and suggest a complete reinterpretation of the velocity variable carried by models. In Section 5, we show that with the new interpretation of the velocity variable, the conservation equations carried by Boussinesq ocean models are almost identical to their non-Boussinesq counterparts in steady state, with the implication that Boussinesq ocean models are actually much more accurate than had hitherto been imagined. The analysis is extended to the non-steady equations in Section 6. The new interpretation of the velocity variable leads to a rather straightforward way of making an ocean model fully non-Boussinesq, as is demonstrated in the follow-on paper by Greatbatch, Lu and Cai(2001). Finally, Section 7 provides a summary and conclusions.

2. The governing equations

The instantaneous conservation equations for mass, for a conservative scalar, C, and for momentum are

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}$$

$$(\rho C)_t + \nabla \cdot (\rho \mathbf{u} C) = \nabla \cdot (\rho \kappa_C \nabla C), \tag{2}$$

$$(\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) + 2\mathbf{\Omega} \times (\rho \mathbf{u}) = -\nabla p - \mathbf{k} g \rho + \nabla \cdot (\mu \nabla \mathbf{u}) + \frac{1}{3}\nabla(\mu \nabla \cdot \mathbf{u}). \tag{3}$$

The terminology here is standard, with μ being the viscosity and κ_C is the molecular diffusivity of property C. It should be noted that C is defined as the mass of tracer per unit mass of fluid (Gill, 1982). C can also be interpreted as the potential temperature. When the Boussinesq approximation is made, (1)-(3) are replaced by

$$\nabla \cdot \mathbf{u} \approx 0, \tag{4}$$

$$C_t + \nabla \cdot (\mathbf{u} C) \approx \nabla \cdot (\kappa_C \nabla C), \tag{5}$$

$$\mathbf{u}_t + \nabla \cdot (\mathbf{u}\mathbf{u}) + 2\mathbf{\Omega} \times \mathbf{u} \approx -\frac{1}{\rho_o}\nabla p - \mathbf{k} g \frac{\rho}{\rho_o} + \frac{1}{\rho_o}\nabla \cdot (\mu \nabla \mathbf{u}). \tag{6}$$

The Boussinesq approximation is really composed of three separate approximations. The first two approximations have long been discussed in the literature, for example by

Spiegel and Veronis (1960), and these two aspects of the Boussinesq approximation are attributed to Boussinesq (1903). The third and potentially most damaging aspect of the Boussinesq approximation has apparently only been realised and published as recently as 1992 by MG.

In the first of the Boussinesq approximations the equation for the conservation of mass, (1), is replaced by the equation for the conservation of volume, (4). The conditions under which this approximation is valid are discussed in detail in Batchelor (1967; Section 3.6), Gill (1982; Section 4.10), and Kundu (1990; Chapter 4).

The second part of the Boussinesq approximation involves replacing the density in (2) and on the left hand side of (3) by a constant representative density, ρ_o. In this way the driving horizontal pressure gradient in (6) is thought to be in error by the replacement of ρ^{-1} with ${\rho_o}^{-1}$. In the ocean, the in situ density varies by no more than 5% and this is the level of approximation associated with this second aspect of the Boussinesq approximation so that traditionally the Boussinesq approximation is thought to cause an error of at most 5% in the velocity vector.

The anelastic approximation (Ogura and Phillips, 1962) is a method for reducing the magnitude of the error associated with the Boussinesq approximation. In the anelastic approximation, the in-situ density on the left-hand side of the conservation equations, (1)-(3), is replaced by a function of pressure (or depth) so that most of the effects of the fluid's compressibility are included. While this would be a distinct improvement over the standard Boussinesq interpretation of the present Boussinesq model equations, it does require substantial modification to the Boussinesq model code. We do not pursue this further because we manage to derive a 100% accurate set of averaged equations that is easier to implement in an ocean model than is the anelastic approximation.

It seems that, prior to the work of MG, all discussions of the Boussinesq approximation have failed to consider the effect of using a divergence free velocity as the advective velocity in the tracer and momentum equations. MG considered the divergence form of the averaged conservation equations, and pointed out that the term associated with the divergence of the Eulerian mean velocity can easily be as large as the diapycnal mixing term. In this paper we will show that the solution to this serious conundrum raised by MG and Davis (1994) is to reinterpret the horizontal "velocity" carried by Boussinesq ocean models as the average horizontal mass flux per unit area normalised by ρ_o. This reinterpretation of what is normally called the "velocity" in Boussinesq ocean models overcomes all three aspects of the Boussinesq approximation in the situation where the flow is steady and geostrophic. In this way we will show that the so-called Boussinesq ocean models have always been more accurate than we had a right to expect:- all we must do is to stop referring to the model "velocity" and instead realise that it is proportional to the average mass flux per unit area.

Another approximation commonly made in ocean models is to replace the real equation of state, $\rho = \rho(S, \theta, p)$, with $\rho = \rho(S, \theta, p_{ref})$, where S and θ are the salinity and potential temperature carried by the model and p_{ref} is a reference pressure that depends only on depth. In this way, most of the dependence of density on pressure is taken into account. Dewar et al. (1998) have argued, nevertheless, that using this simplified equation of state leads to significant error and that the full equation of state should be used to compute density for use in the hydrostatic equation. We return to this issue in Section 5.

3. The Boussinesq Conundrum

We now revisit the concern raised by MG that arises in applying the Boussinesq approximation to the averaged tracer conservation equation. We derive this result by first writing the equations (1)- (3) in the advective form

$$\begin{aligned} \nabla \cdot \mathbf{u} &= -\rho^{-1}(\rho_t + \mathbf{u} \cdot \nabla \rho), & (7)\\ C_t + \mathbf{u} \cdot \nabla C &= \rho^{-1} \nabla \cdot (\rho \kappa_C \nabla C) = d_C, & (8)\\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} &= -\rho^{-1} \nabla p - \mathbf{k} g + \mathbf{d_u}. & (9) \end{aligned}$$

The molecular viscosity terms in (9) are given by $\mathbf{d_u} = \rho^{-1} \nabla \cdot (\mu \nabla \mathbf{u}) + \frac{1}{3}\rho^{-1} \nabla (\mu \nabla \cdot \mathbf{u})$. Following MG and Davis (1994) these instantaneous equations are ensemble averaged (or temporally averaged with a long averaging time) finding (still without any approximations whatsoever)

$$\begin{aligned} \nabla \cdot \overline{\mathbf{u}} &= -\overline{\rho}^{-1}(\overline{\rho}_t + \overline{\mathbf{u}} \cdot \nabla \overline{\rho}) - \overline{\rho}^{-1} \nabla \cdot (\overline{\mathbf{u}'\rho'}), & (10)\\ \overline{C}_t + \overline{\mathbf{u}} \cdot \nabla \overline{C} &= \overline{d_C} - \nabla \cdot (\overline{\mathbf{u}'C'}) + \overline{C' \nabla \cdot \mathbf{u}'}, & (11)\\ \overline{\mathbf{u}}_t + \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{u}} + 2\mathbf{\Omega} \times \overline{\mathbf{u}} &= -\overline{\rho^{-1} \nabla p} - \mathbf{k} g + \overline{\mathbf{d_u}} - \nabla \cdot (\overline{\mathbf{u}'\mathbf{u}'}) + \overline{\mathbf{u}' \nabla \cdot \mathbf{u}'}. & (12) \end{aligned}$$

The so-called "incompressibility" condition is the assumption that the right-hand side of (10) can be ignored compared with the individual components of $\nabla \cdot \overline{\mathbf{u}}$ such as $\overline{w}_z$. The classic Boussinesq approximation replaces $-\overline{\rho^{-1} \nabla p}$ in (12) with $-{\rho_o}^{-1} \nabla_H \overline{p} - \overline{\rho}^{-1} \overline{p}_z \mathbf{k}$ and ignores the last terms in (11) and (12). It is usually thought this involves say a 5% error in the horizontal pressure gradient term in the deep ocean. Note that while the classic Boussinesq approximation replaces in-situ density with the reference density in the instantaneous tracer equation, (2), in our Reynolds-averaged tracer equation, (11), this is equivalent to ignoring the $\overline{C' \nabla \cdot \mathbf{u}'}$ term on the right-hand side. All these changes are commonly referred to as the Boussinesq approximation.

In the appendix we confirm MG's results that (i), the average of the molecular diffusion term, $\overline{d_C}$, holds no surprises and so can either be ignored or absorbed into the turbulent mixing term, and that, (ii), $\overline{C' \nabla \cdot \mathbf{u}'}$ is small enough to be ignored. Hence (11) reduces to the regular advective form of a conservation statement, namely

$$\overline{C}_t + \overline{\mathbf{u}} \cdot \nabla \overline{C} = -\nabla \cdot (\overline{\mathbf{u}'C'}). \quad (13)$$

This seems to confirm the accuracy of the tracer equation in the Boussinesq model equations (we dispute this below). MG, nevertheless, showed that the divergence form of the Boussinesq tracer equation is in error by as much as the diapycnal mixing term in that equation. Diapycnal mixing is known to be an important and often dominant physical process in the deep ocean. MG's analysis showed that if we knew exactly the Eulerian-mean velocity, $\overline{\mathbf{u}}$, and used this velocity to evaluate the tracer budget using the divergence form of the Boussinesq tracer equation, then serious error would result. This point is also made by Davis(1994).

This causes us to now question the traditional interpretation of the velocity in ocean models as being the Eulerian-mean velocity, because we do not want to believe that Boussinesq ocean models are so seriously in error as to be incapable of representing the diapycnal advective-diffusive balance. We now use the method of Lu (2001) to examine more closely the error involved in a Boussinesq ocean model when we make the usual (but we now believe incorrect) assumption that the horizontal momentum equations are prognostic equations for the Eulerian-mean horizontal velocity. The model's velocity vector is therefore defined as $\hat{\mathbf{u}} = \overline{\mathbf{u}} + \delta w \mathbf{k}$, where here (in contrast to Lu) $\overline{\mathbf{u}}$ is the Eulerian mean velocity. The model's vertical velocity is different from the Eulerian mean because of the requirement that the model's three-dimensional velocity field be divergence free. $\delta w \mathbf{k}$ is therefore chosen, following Lu(2001), so that $\nabla \cdot \hat{\mathbf{u}} = 0$ everywhere and $\delta w = 0$ at the ocean bottom so that

$$\delta w = -\int_{-h}^{z} \nabla \cdot \overline{\mathbf{u}} dz. \tag{14}$$

Writing (13) in terms of the model's velocity, $\hat{\mathbf{u}}$, (ignoring $\overline{d_C}$ and $\overline{C' \nabla \cdot \mathbf{u}'}$) we obtain

$$\frac{\partial \overline{C}}{\partial t} + \hat{\mathbf{u}} \cdot \nabla \overline{C} = \frac{\partial \overline{C}}{\partial t} + \nabla \cdot [\hat{\mathbf{u}}\ \overline{C}] = -\nabla \cdot [\overline{\mathbf{u}' C'}] + \delta w \frac{\partial \overline{C}}{\partial z}. \tag{15}$$

It follows from (15) that the usual assumption that the horizontal velocity carried by an ocean model is the Eulerian-mean horizontal velocity results in the error term

$\delta w\ \overline{C}_z$ in the tracer conservation statement in such a model. MG showed that the dominant contribution to $\nabla\cdot\overline{\mathbf{u}}$ comes from the compressible nature of seawater according to

$$\nabla\cdot\overline{\mathbf{u}} \approx -\overline{\gamma}(\overline{p}_t + \overline{\mathbf{u}}\cdot\nabla\overline{p}) \approx g\overline{w}/c^2 \tag{16}$$

where γ is the adiabatic compressibility of seawater, equal to $(\rho c^2)^{-1}$ where c is the sound speed, and the right-hand side of (16) has been written using the hydrostatic approximation. Assuming a typical mean vertical velocity, $\overline{w}$, of 2×10^{-6} m s^{-1}, gives an estimate for $\nabla\cdot\overline{\mathbf{u}}$ of 10^{-11} s^{-1}. Using a depth scale of 1000m this implies, from (14), that $\delta w \approx 10^{-8}$ m s^{-1}. Putting $C = S$ (salinity), we estimate $\overline{S}_z$ to be 10^{-3}m^{-1}. Hence $\delta w\ \overline{S}_z$ has magnitude 10^{-11}s^{-1}, and so is the same order of magnitude as the diapycnal mixing term (assuming a diapycnal diffusivity of 10^{-5} m^2 s^{-1}). This analysis of errors in ocean models is more relevant than the one advanced by MG because it recognises that the velocity variable carried by a Boussinesq ocean model is divergence free, and so is $\hat{\mathbf{u}}$, not $\overline{\mathbf{u}}$.

It is important to realise that the extra source term, $\delta w\ \overline{C}_z$, that appears in (15) is a leading order term that upsets the proper balance between diapycnal advection and diapycnal diffusion in this equation. Davis(1994) emphasised that this missing term can lead to a 100% mis-estimation of the diapycnal diffusivity in inverse studies of the ocean circulation (assuming that the Eulerian mean velocity is available to the inversion). The form of the term as we have written it, namely $\delta w\ \overline{C}_z$, would seem to indicate that the term is inherently advective in nature. This is not the case. As Davis showed, this term could equally well be regarded as a non-conservative alteration to the diapycnal diffusivity. (The fundamentally non-conservative nature of this term is apparent from our analysis of the unsteady equations in Section 6, following equation (43).) We have shown here that if one interprets the horizontal velocity as the Eulerian-mean velocity, then this Boussinesq conundrum source, $\delta w\ \overline{C}_z$, is far too large to ignore, amounting to as much as the effects of diapycnal mixing. Unless we can find a way around this

Boussinesq conundrum, we oceanographers have no right to use so-called Boussinesq models for any application where diapycnal mixing is significant, for example, in climate modelling.

It is also important to note that (15) has been written in equivalent advective and divergence forms. It follows that the error identified by MG is not confined to the divergence form of the conservation statement, but is also a feature of the advective form. In particular, the use of the divergence free velocity $\hat{\mathbf{u}}$ to advect tracer leads to an error, represented by the $\delta w\, \overline{C}_z$ term. If the advection velocity were the full Eulerian mean velocity, $\overline{\mathbf{u}}$, then, (13) shows that there is then no serious error. The problem is that in models, the advective velocity is not the Eulerian mean velocity, because, in contrast to the Eulerian mean velocity, the model velocity is divergence free. It follows that the third part of the Boussinesq approximation, identified originally by MG as a problem with the divergence form of the averaged Boussinesq tracer equation, has its counterpart when the equation is written in advective form, and arises because the Boussinesq velocity is no longer the Eulerian mean velocity.

Since the dominant term in the divergence of the mean velocity is $g\overline{w}/c^2$, the vertical velocity difference $|\delta w| = |\hat{\mathbf{u}} - \overline{\mathbf{u}}|$ scales as $g\overline{w}h/c^2$ (from (14)) where here h is the vertical distance over which the vertical velocity remains correlated. One of the terms on the left-hand side of (15) is $\overline{w}\, \overline{C}_z$ so the error term, $\delta w\, \overline{C}_z$ as a fraction of $\overline{w}\, \overline{C}_z$ is

$$(\delta w\, \overline{C}_z)/(\overline{w}\, \overline{C}_z) \approx gh/c^2 \approx 0.01 \qquad (17)$$

where h has been taken to be 2000m and the sound speed 1500 m s^{-1}. Hence one might be tempted to conclude that the error term in (15) is no more than 1% of the other terms in the equation. However, we know that the vertical and horizontal advection terms on the left-hand side of (15) almost self-cancel, leaving a much smaller advection that balances the divergence of the turbulent flux on the right. Hence the relevant measure of the error term in (15) is the ratio of $\delta w\, \overline{C}_z$ to the diapycnal mixing term, that is

$$(\delta w\, \overline{C}_z)/(\overline{e}\, \overline{C}_z) \approx (gh/c^2)(\overline{w}/\overline{e}) \approx 0.01(\overline{w}/\overline{e}) \tag{18}$$

where $\overline{e} \approx \kappa/h$ has the dimensions of velocity and will be referred to as the diapycnal velocity, and κ is the diapycnal diffusivity. The reason why the diapycnal advection and diffusion terms are the relevant metrics with which to compare the neglected Boussinesq terms is that these diapycnal advection and diffusion terms are two of the primary terms representing real, physical processes in the conservation equation (the others being isopycnal advection and diffusion). By contrast, the near cancellation between horizontal and vertical advection in a Cartesian model says nothing about any physical process at work in the ocean, but merely reflects the use of a Cartesian, as distinct from isopycnal (or adiabatic), coordinate system.

Since the diapycnal velocity is expected to range between 10^{-8} m s^{-1} and 10^{-7} m s^{-1}, while the mean vertical velocity is typically 10^{-6} m s^{-1}, it is clear that the ratio (18) can often be as large as unity implying that the error term in the mean tracer conservation equation due to the Boussinesq approximation is often as large as the diapycnal mixing term in this equation. This error analysis demonstrates that it is the smallness of the diapycnal velocity due to the layered, nearly adiabatic, nature of the ocean that elevates the relative importance of the error involved in the Boussinesq approximation. Regions of the ocean as large as the subtropical gyres have the mean vertical velocity, $\overline{w}$, of one sign and in these regions, the missing Boussinesq error term in (15), $\delta w\, \overline{C}_z$, can be as large as the real effect of diapycnal mixing processes.

Here we summarise the work so far in order to provide context for what follows. We began by introducing MG's amazing result that the divergence form of the mean tracer equation, based on the three-dimensional Eulerian-mean velocity, is in error by as much as the magnitude of the diapycnal mixing term. MG clung to the hope that with ocean models making the false assertion that $\nabla \cdot \overline{\mathbf{u}} = 0$ in two separate places, they may not suffer the full effect of this Boussinesq error. This false hope was debunked by

Davis (1994) who demonstrated that the full strength of the Boussinesq error remained after integrating the conservation equations over large ocean volumes, and he showed that this error was as much as the effects of diapycnal mixing. In this section we have used the method of Lu (2001) to focus on the error that the Boussinesq models contain in their tracer equations, all the while using the common assumption that the model's horizontal velocity is the Eulerian-mean velocity. Again, the error is as large as the diapycnal mixing term, and this is true in both the divergence and advective forms of the model's tracer equation. We describe these results as a conundrum because we really do not believe that present Boussinesq ocean models actually contain this error, and yet a solution to this puzzle has not emerged in the years from the publication of MG and Davis (1994) to the present. In this paper we advance a solution to this conundrum:- the horizontal velocity in ocean models (including in Boussinesq ocean models) is not, and never was, the Eulerian-mean velocity but is actually proportional to the horizontal mass flux per unit area.

4. Density-weighted averaging and the average mass flux

The new approach to be introduced here is based on the density-weighted averaging of equations (1)-(3) in a fixed coordinate system. This method is also called Favre-averaging after Favre (1965 a, b), and goes back at least to Hesselberg (1926). We define

$$\overline{\mathbf{u}}^{\rho} = \overline{\rho\mathbf{u}}/\overline{\rho}, \quad \overline{C}^{\rho} = \overline{\rho C}/\overline{\rho}, \quad \mathbf{u}'' = \mathbf{u} - \overline{\mathbf{u}}^{\rho} \quad \text{and} \quad C'' = C - \overline{C}^{\rho}. \tag{19}$$

where $\overline{\mathbf{u}}^{\rho}$ and $\overline{C}^{\rho}$ are the density-weighted averages of velocity and tracer concentration, and it follows that $\overline{\rho\mathbf{u}''} = \overline{\rho C''} = 0$. Averaging the instantaneous conservation equations (1)-(3) in this way leads to

$$\overline{\rho}_t + \nabla\cdot(\overline{\rho}\ \overline{\mathbf{u}}^{\rho}) = 0, \tag{20}$$

$$(\overline{\rho}\overline{C}^{\rho})_t + \nabla\cdot(\overline{\rho}\ \overline{\mathbf{u}}^{\rho}\ \overline{C}^{\rho}) = -\nabla\cdot(\overline{\rho\mathbf{u}''C''}) + \nabla\cdot(\overline{\rho\kappa_C\nabla C}), \tag{21}$$

$$(\overline{\rho}\,\overline{\mathbf{u}}^{\rho})_t + \nabla\cdot(\overline{\rho}\,\overline{\mathbf{u}}^{\rho}\,\overline{\mathbf{u}}^{\rho}) + 2\boldsymbol{\Omega}\times(\overline{\rho}\,\overline{\mathbf{u}}^{\rho}) = -\nabla\overline{p} - \mathbf{k}g\overline{\rho} - \nabla\cdot(\overline{\rho\mathbf{u}''\mathbf{u}''}) + \nabla\cdot(\overline{\mu\nabla\mathbf{u}}) + \frac{1}{3}\nabla(\overline{\mu\nabla\cdot\mathbf{u}}). \tag{22}$$

Noting that on the left-hand side of these equations, $\overline{\mathbf{u}}^{\rho}$ appears mostly in the form of the product $\overline{\rho}\,\overline{\mathbf{u}}^{\rho}$, we are motivated to write (20)-(22) in terms of a modified mean "velocity" vector

$$\overline{\tilde{\mathbf{u}}} = \overline{\rho}\,\overline{\mathbf{u}}^{\rho}/\rho_o = \overline{\rho\mathbf{u}}/\rho_o. \tag{23}$$

$\overline{\tilde{\mathbf{u}}}$ is simply a scaled version of the average mass flux per unit area, since ρ_o is a constant reference density, say 1030 kg m^{-3}. (20) - (22) then become

$$(\overline{\rho}/\rho_o)_t + \nabla\cdot\overline{\tilde{\mathbf{u}}} = 0, \tag{24}$$

$$(\frac{\overline{\rho}}{\rho_o}\overline{C}^{\rho})_t + \nabla\cdot(\overline{\tilde{\mathbf{u}}}\,\overline{C}^{\rho}) = \nabla\cdot(\mathbf{K}\nabla\overline{C}^{\rho}), \tag{25}$$

$$\overline{\tilde{\mathbf{u}}}_t + \nabla\cdot(\frac{\rho_o}{\overline{\rho}}\overline{\tilde{\mathbf{u}}}\,\overline{\tilde{\mathbf{u}}}) + 2\boldsymbol{\Omega}\times\overline{\tilde{\mathbf{u}}} = -\frac{1}{\rho_o}\nabla\overline{p} - \mathbf{k}g\frac{\overline{\rho}}{\rho_o} + \nabla\cdot(\mathbf{A}\nabla\frac{\rho_o}{\overline{\rho}}\overline{\tilde{\mathbf{u}}}). \tag{26}$$

Note that the molecular flux terms in (21) and (22) (involving κ_C and μ) have been absorbed into the turbulent fluxes, and that the turbulent fluxes have been parameterized using a Fickian approach, as is traditional (noting that the diffusion tensors **K** and **A** may have both symmetric and antisymmetric parts).

These are the fully non-Boussinesq conservation equations, written in terms of our new velocity variable which is proportional to the average mass flux per unit area. The Boussinesq approximation consists of replacing $\overline{\rho}$ with ρ_o everywhere except in the vertical gravitational acceleration term. If we perform this Boussinesq replacement procedure on (24) - (26), we obtain our version of the Boussinesq conservation equations,

$$\nabla\cdot\overline{\tilde{\mathbf{u}}} \approx 0, \tag{27}$$

$$\overline{C}^{\rho}_t + \nabla\cdot(\overline{\tilde{\mathbf{u}}}\,\overline{C}^{\rho}) \approx \nabla\cdot(\mathbf{K}\nabla\overline{C}^{\rho}), \tag{28}$$

$$\overline{\tilde{\mathbf{u}}}_t + \nabla\cdot(\overline{\tilde{\mathbf{u}}}\,\overline{\tilde{\mathbf{u}}}) + 2\boldsymbol{\Omega}\times\overline{\tilde{\mathbf{u}}} \approx -\frac{1}{\rho_o}\nabla\overline{p} - \mathbf{k}g\frac{\overline{\rho}}{\rho_o} + \nabla\cdot(\mathbf{A}\nabla\overline{\tilde{\mathbf{u}}}). \tag{29}$$

Notice that these equations correspond exactly to the instantaneous Boussinesq equations, (4) - (6), but here we interpret the variables (particularly the velocity) in a special way to ensure, as we shall show, that (27)-(29) are much more accurate than the average of (4)-(6).

5. The steady, geostrophic hydrostatic equations are fully non-Boussinesq

We begin by noting that if the mean fields are in a steady state (that is if $\overline{\rho}_t$ and $\overline{C}^{\rho}_t$ are zero), the fully non-Boussinesq continuity and tracer equations (i.e. (24) and (25), respectively) can be written very simply as $\nabla \cdot \overline{\tilde{\mathbf{u}}} = 0$ and $\nabla \cdot (\overline{\tilde{\mathbf{u}}}\ \overline{C}^{\rho}) = \nabla \cdot (\mathbf{K}\nabla\overline{C}^{\rho})$. These conservation equations are exactly the ones used by numerical ocean models, and contrary to common assumption, there are no error terms here of up to 5 % magnitude, as normally associated with the Boussinesq approximation, nor are there errors of order 30 % or more, as implied by MG. Furthermore, under the geostrophic and hydrostatic balance, the momentum equation also holds without error. In summary, when the ocean is statistically steady, geostrophic and hydrostatic, the continuity, tracer and momentum equations are, without any Boussinesq error,

$$\nabla \cdot \overline{\tilde{\mathbf{u}}} = 0; \quad \nabla \cdot (\overline{\tilde{\mathbf{u}}}\ \overline{C}^{\rho}) = \nabla \cdot (\mathbf{K}\nabla\overline{C}^{\rho}); \quad 2\mathbf{\Omega} \times \overline{\tilde{\mathbf{u}}}_H = -\frac{1}{\rho_o}\nabla_H\overline{p}; \quad \text{and} \quad \overline{p}_z = -g\overline{\rho} \qquad (30)$$

where $\overline{\tilde{\mathbf{u}}}_H$ is the horizontal component of $\overline{\tilde{\mathbf{u}}}$ and ∇_H is the horizontal gradient operator. This implies that, subject to the geostrophic restriction, when the present generation of hydrostatic ocean models reach a steady state, they are in fact fully non-Boussinesq and so do not suffer the errors of 5 % or more associated with the Boussinesq approximation.

We wish to emphasize that the equations in (30) have been derived without the need to make the Boussinesq approximation and yet they are exactly the same conservation equations as are used in Boussinesq numerical models of the ocean circulation. Certainly it *appears* that the Boussinesq approximation has been made in (30) because (i) $\overline{\rho}$ is not present inside the divergence terms $\nabla \cdot \overline{\tilde{\mathbf{u}}}$ and $\nabla \cdot (\overline{\tilde{\mathbf{u}}}\ \overline{C}^{\rho})$, and (ii) there is a constant reference density in the $-\nabla_H \overline{p}/\rho_o$ term. However we have avoided having to make the Boussinesq approximation by redefining the velocity vector as being the average mass flux per unit area (and then dividing by ρ_o to give it the dimensions of velocity).

There are two remarks that need to be made in regard to the hydrostatic balance in (30), $\overline{p}_z = -g\overline{\rho}$, and these remarks remain pertinent in the more general situation where the flow is unsteady and the momentum equation is not simply taken to be the geostrophic balance. The first remark is the point made by Dewar et al. (1998) that $\overline{\rho}$ should be allowed to respond to the changing pressure at fixed depth. While the past practice in this regard is not actually part of a Boussinesq approximation (since the appearance of the in situ density in the vertical momentum equation is the only place in the Boussinesq approximation procedure where the in situ density is not replaced by ρ_o) it has led to errors of similar magnitude to that usually associated with the Boussinesq approximation.

The second remark relates to our ability to evaluate $\overline{\rho}$ given the fact that the model is assumed to carry the density-weighted salinity and potential temperature, $\overline{S}^{\rho}$ and $\overline{\theta}^{\rho}$. In hydrostatic ocean models, the hydrostatic equation is vertically integrated to yield the pressure whose horizontal gradient appears in the horizontal momentum equations. An estimate of the error due to our inability to exactly determine $\overline{\rho}$ can be gained by examining the thermal wind equation which can be found from (30), namely,

$$2\boldsymbol{\Omega} \times (\overline{\tilde{\mathbf{u}}}_H)_z = \frac{g}{\rho_o} \nabla_H \overline{\rho}. \qquad (31)$$

Here one needs the horizontal gradient of $\overline{\rho}$, and with a linear equation of state, this can be obtained from a knowledge of $\overline{S}$ and $\overline{\theta}$ because $\nabla_H \overline{\rho} = \nabla_H \rho(\overline{S}, \overline{\theta}, \overline{p})$. Before commenting on the influence of the non-linear nature of the equation of state, we need to consider the fact that the model variables are $\overline{S}^{\rho}$ and $\overline{\theta}^{\rho}$ rather than being the Eulerian-mean salinity and potential temperature. The difference between these salinities and potential temperatures makes the following difference to the horizontal density gradient,

$$\begin{aligned} \nabla_H \rho(\overline{S}^\rho, \overline{\theta}^\rho, \overline{p}) - \nabla_H \rho(\overline{S}, \overline{\theta}, \overline{p}) &\approx \rho_o(\beta \nabla_H [\overline{S}^\rho - \overline{S}] - \alpha \nabla_H [\overline{\theta}^\rho - \overline{\theta}]) \\ &= \rho_o(\beta \nabla_H \left[\frac{\overline{\rho' S'}}{\overline{\rho}} \right] - \alpha \nabla_H \left[\frac{\overline{\rho' \theta'}}{\overline{\rho}} \right]) \\ &\approx \frac{1}{\rho_o} \nabla_H \left[\overline{(\rho')^2} \right] . \end{aligned} \tag{32}$$

This error in estimating the horizontal gradient of in-situ density is only 0.1 % of a typical horizontal density gradient, and when integrating the thermal wind equation over the whole water column, with $\nabla_H \left[\overline{(\rho')^2} \right]$ being estimated using a density variation at a fixed location of 0.15 kg m^{-3} varying horizontally by its own magnitude in a distance of 10^6 m, the error in the horizontal velocity is only 10^{-5} m s^{-1}. This shows that the difference between $\nabla_H \rho(\overline{S}^\rho, \overline{\theta}^\rho, \overline{p})$ and $\nabla_H \rho(\overline{S}, \overline{\theta}, \overline{p})$ is of no significant consequence.

The nonlinear nature of the equation of state of seawater is a separate reason why a model cannot exactly determine $\nabla_H \overline{\rho}$. Following McDougall and McIntosh (1996), the difference between $\nabla_H \rho(\overline{S}, \overline{\theta}, \overline{p})$ and $\nabla_H \overline{\rho}$ is of order $-\frac{1}{2} \rho_o \nabla_H \left[\frac{\partial \alpha}{\partial \theta} \overline{(\theta')^2} \right]$ where $\frac{\partial \alpha}{\partial \theta}$ is the variation of the thermal expansion coefficient with potential temperature. This contribution to $\nabla_H \overline{\rho}$ is estimated to be two orders of magnitude larger than the estimate obtained from (32). As such, this effect can cause an error in the horizontal velocity of order 10^{-3} m s^{-1}. An error of this magnitude would not be trivial in an ocean model if it was a persistent error (for example if an error of this magnitude occurred all the way along a zonal average). However this error enters as a horizontal divergence and so it does not lead to any persistent effects. This error in determining $\nabla_H \overline{\rho}$ and $\nabla_H \overline{p}$ has been present in all ocean modeling to date and it has never been recognized as a problem. Neither can we envisage that this effect will cause significant inaccuracies and so we recommend that the Eulerian-mean density, $\overline{\rho}$, that appears in the vertical component of the momentum equation in (26), can be evaluated using the model variables and the equation of state as $\rho(\overline{S}^\rho, \overline{\theta}^\rho, \overline{p})$.

6. The unsteady equations

The full continuity equation, (24), contains the temporal derivative of mean in-situ density in the term $(\overline{\rho}/\rho_o)_t$, which, in unsteady situations, is non-zero. If, as an example, we assume a warming or cooling at the rate of 1 °C in 30 years, and that the density change is dominated by the temperature change, then

$$\nabla \cdot \overline{\tilde{\mathbf{u}}} = -(\overline{\rho}/\rho_o)_t \approx 2 \times 10^{-13}\mathrm{s}^{-1}. \tag{33}$$

This is fifty times smaller than the estimate for the divergence of the Eulerian mean velocity of 10^{-11} s^{-1} given by MG. On the other hand, on the seasonal time scale, if we assume that temperature changes by 10°C in 100 days, and that the density change is again dominated by the temperature change, we obtain $(\overline{\rho}/\rho_o)_t \approx 10^{-10}\mathrm{s}^{-1}$. For mesoscale eddies, we estimate $(\overline{\rho}/\rho_o)_t \approx 10^{-9}\mathrm{s}^{-1}$. Both these values are considerably bigger than MG's estimate for $\nabla \cdot \overline{\mathbf{u}}$ and raise a question concerning the accuracy of the unsteady Boussinesq equations (27)-(29) in comparison with their non-Boussinesq counterparts (24)-(26).

To address this issue, we again use a technique based on the method of Lu(2001). We begin by noting that the continuity equation is

$$(\overline{\rho}/\rho_o)_t + \nabla \cdot \overline{\tilde{\mathbf{u}}} = 0. \tag{34}$$

We now define a new velocity variable by

$$\hat{\mathbf{u}} = \overline{\tilde{\mathbf{u}}} + \delta w \mathbf{k} \tag{35}$$

and choose δw so that $\nabla \cdot \hat{\mathbf{u}} = 0$ everywhere, and $\delta w = 0$ at the ocean bottom. (Note that $\hat{\mathbf{u}}$ in (35) is different from $\hat{\mathbf{u}}$ in (15).) It follows that this time

$$\delta w = \int_{-h}^{z} \frac{\partial}{\partial t}[\frac{\overline{\rho}}{\rho_o}]dz. \tag{36}$$

Let us consider what happens when $(\overline{\rho}/\rho_o)_t$ is governed primarily by advective processes, as we expect to be the case in eddy-resolving models. It can then be shown that $\delta w/\overline{\tilde{w}}$ is of order $\Delta\rho/\rho_o \approx$ a few %, implying that to the same level of approximation, (24) can be replaced by (27), despite the seemingly large estimate for the magnitude of $(\overline{\rho}/\rho_o)_t$ noted above. For example, if vertical advection is dominant, as in linear dynamics, then

$$(\overline{\rho}/\rho_o)_t \approx \overline{\tilde{w}}\overline{\rho}_z/\rho_o, \tag{37}$$

and a simple scale analysis using (36) shows that

$$\delta w/\overline{\tilde{w}} \approx \frac{H\overline{\rho}_z}{\rho_o} \approx \frac{\Delta\rho}{\rho_o}. \tag{38}$$

We now turn to the tracer equation (25) and note that the momentum equation can be treated similarly. Writing (25) in terms of the divergence free velocity $\hat{\mathbf{u}}$, we obtain

$$\overline{C}^{\rho}{}_t + \nabla\cdot(\hat{\mathbf{u}}\;\overline{C}^{\rho}) = (\frac{(\rho_o-\overline{\rho})}{\rho_o}\overline{C}^{\rho})_t + \frac{\partial(\delta w\overline{C}^{\rho})}{\partial z} + \nabla\cdot(\mathbf{K}\nabla\overline{C}^{\rho}). \tag{39}$$

Using the continuity equation, (34), this can be written as

$$\begin{aligned}&\left(\overline{C}^{\rho} - C_R\right)_t + \nabla\cdot\left(\hat{\mathbf{u}}\;(\overline{C}^{\rho} - C_R)\right)\\ &\qquad = \left(\frac{(\rho_o-\overline{\rho})}{\rho_o}(\overline{C}^{\rho} - C_R)\right)_t + \frac{\partial(\delta w(\overline{C}^{\rho} - C_R))}{\partial z} + \nabla\cdot(\mathbf{K}\nabla\overline{C}^{\rho}).\end{aligned} \tag{40}$$

where C_R is a constant, reference value of $\overline{C}^{\rho}$. We now follow Davis(1994) and vertically integrate (40) from the bottom, $z=-h$, to the (time-varying) height of the iso-surface $\overline{C}^{\rho} = C_R$ of the tracer $\overline{C}^{\rho}$. Since $\delta w = 0$ at the bottom, and since $\overline{C}^{\rho} = C_R$ at the top of the range of integration, it follows that the penultimate term in (40) integrates to zero and the equation becomes

$$\begin{aligned}&\left(\int_{-h}^{z(C_R)}(\overline{C}^{\rho} - C_R)\;dz\right)_t + \nabla_H\cdot\left(\int_{-h}^{z(C_R)}\overline{\tilde{\mathbf{u}}}_H\;(\overline{C}^{\rho} - C_R)\;dz\right)\\ &\qquad = \left(\int_{-h}^{z(C_R)}\frac{(\rho_o-\overline{\rho})}{\rho_o}(\overline{C}^{\rho} - C_R)\;dz\right)_t + \int_{-h}^{z(C_R)}\nabla\cdot(\mathbf{K}\nabla\overline{C}^{\rho})\;dz,\end{aligned} \tag{41}$$

where ∇_H is the horizontal gradient operator and we have used the fact that $\hat{\mathbf{u}}_H = \overline{\tilde{\mathbf{u}}}_H$, the subscript H denoting "horizontal component". The Boussinesq equivalent of this equation can be deduced from (28) and is

$$\left(\int_{-h}^{z(C_R)} (\overline{C}^\rho - C_R)\ dz\right)_t + \nabla_H \cdot \left(\int_{-h}^{z_R)} \tilde{\overline{\mathbf{u}}}_H\ (\overline{C}^\rho - C_R)\ dz\right)$$
$$\approx \int_{-h}^{z(C_R)} \nabla \cdot (\mathbf{K}\nabla\overline{C}^\rho)\ dz. \qquad (42)$$

To make our point, we assume for simplicity that we have a linear equation of state and regard C as potential density, although this assumption is not essential to our argument. We now note that the diapycnal flux of tracer across the surface $\overline{C}^\rho = C_R$ due to diapycnal mixing is obtained from $\int_{-h}^{z(C_R)} \nabla \cdot (\mathbf{K}\nabla\overline{C}^\rho)\ dz$ in both Boussinesq and non-Boussinesq versions. Further, the Boussinesq version differs from the non-Boussinesq version only by the first term on the right hand side of (41). In particular, there is no contribution to the Boussinesq error from δw. Furthermore, the Boussinesq error appears as a local time derivative, and hence, averages to zero under long time-averaging. It follows that there is no systematic error in the diapycnal transport of tracer from the Boussinesq approximation. We stress, however, that this result has been obtained using our interpretation of the variables in the Boussinesq tracer equation, as written in (28). In particular, the horizontal velocity variable is interpreted as the horizontal component of $\tilde{\overline{\mathbf{u}}}$, not as the horizontal component of the Eulerian mean velocity $\overline{\mathbf{u}}$ (the vertical velocity being determined from the assumption of divergence-free flow).

The importance of interpreting the horizontal velocity variable as $\tilde{\overline{\mathbf{u}}}_H$ can be understood by applying the same analysis to (15), in which the horizontal velocity variable is interpreted as the Eulerian mean horizontal velocity, $\overline{\mathbf{u}}_H$. In this case, the equivalent of (41) is

$$\left(\int_{-h}^{z(C_R)} (\overline{C} - C_R)\ dz\right)_t + \nabla_H \cdot \left(\int_{-h}^{z(C_R)} \overline{\mathbf{u}}_H\ (\overline{C} - C_R)\ dz\right)$$
$$= \int_{-h}^{z(C_R)} \nabla \cdot (\mathbf{K}\nabla\overline{C})\ dz + \int_{-h}^{z(C_R)} \delta w \frac{\partial \overline{C}}{\partial z} dz \qquad (43)$$

where the vertical integral is between the ocean bottom, $z = -h$, and the time-varying surface $\overline{C} = C_R$. The Boussinesq correction term now appears in terms of δw, but unlike in (41), it is present in the vertically integrated equation, and does not average to zero under long time averaging, implying the possibility of systematic error in this case. Indeed, we believe systematic error is present since there are parts of the ocean (e.g. the interior of the subtropical and subpolar gyres, or in the equatorial region) where both δw and $\overline{C}_z$ are persistently of one sign. (Note that δw is given here by (14), and that $\nabla \cdot \overline{\mathbf{u}}$ is dominated by the contribution from the mean vertical velocity, $\overline{w}$, as discussed following (16).) It follows that in such regions, the error term $\int_{-h}^{z(C_R)} \delta w \frac{\partial \overline{C}}{\partial z} dz$ does not average to zero under long time averaging and will result in systematic error.

Finally in this section, we note that there are only four locations where the factor $\overline{\rho}/\rho_o$ (or its reciprocal) needs to be added to the Boussinesq conservation equations (27)-(29) to make them fully non-Boussinesq. It follows that it should be relatively simple to modify existing Boussinesq code to make it fully non-Boussinesq, as demonstrated by Greatbatch et al. (2001) in the case of the POP model (POP is "Parallel Ocean Program", the parallel version of the GFDL ocean model developed at Los Alamos). Greatbatch et al. describe the detailed changes that are needed to the model numerics as well as some results that illustrate the benefits of having a fully non-Boussinesq model. As shown in that paper, the overhead in additional cpu requirement is modest in the case of POP.

7. Conclusions

McDougall and Garrett (1992) and Davis (1994) warned that the use of the Boussinesq approximation can lead to errors in the averaged tracer conservation equation that are the same order as the diapycnal mixing term. This warning was based on interpreting the velocity as the Reynolds- or Eulerian-averaged velocity. In section 3 we extended their analysis to specifically apply to the present generation of so-called Boussinesq ocean models in which the model velocity is required to be divergence free.

We noted that the relative magnitude of the error is enhanced by the large cancellation between the horizontal and vertical advection terms in the model's tracer equation on sloping isopycnal surfaces, leaving a much smaller advection (of Boussinesq magnitude with the conventional interpretation of the model's variables) to balance the diapycnal mixing. This served to confirm the serious conundrum which has plagued our field since 1992. On the face of it, the Boussinesq conundrum means that we oceanographers cannot use Boussinesq models for any application in which diaycnal mixing is important, for example for climate studies. While for the past nine years we have thought it very unlikely that all Boussinesq ocean models have been in error by as much as the magnitude of the diapycnal mixing term, a solution to this conundrum has not been proposed until now.

In the present paper we show that the vector we call "velocity" in an ocean model is not, and never was, the fluid velocity. Rather the horizontal velocity is (and always was) proportional to the horizontal mass flux per unit area. By interpreting the model horizontal "velocity" in this fashion, the large error disappears from the model's tracer equation and the conundrum is overcome. When present so-called Boussinesq ocean models achieve a steady state, we have shown that they are almost completely non-Boussinesq. We also showed that with the above interpretation of the model horizontal "velocity", the unsteady equations carried by Boussinesq ocean models contain no systematic error in the diapycnal advective/diffusive balance due to the Boussinesq approximation.

The emphasis we have placed on the interpretation of the horizontal velocity vector carried by Boussinesq ocean models is motivated by the procedure adopted in hydrostatic ocean models where the vertical velocity is diagnosed from the horizontal velocity using the requirement that the three-dimensional velocity vector be divergence free. However, there is nothing in our analysis that restricts it to the hydrostatic case. Non-hydrostatic models enforce the same zero-divergence of the velocity field, although the full vertical momentum equation is carried to compute the vertical velocity. Hence, the same analyses of the Boussinesq error in the averaged tracer equations, associated with neglecting the velocity divergence, applies to both the hydrostatic and non-hydrostatic cases.

Throughout the analysis in Sections 5 and 6, we have interpreted the tracer variable as the density-weighted averaged. It should be noted that similar conclusions apply if the tracer variable is interpreted instead as the conventional Reynolds average (the effect of doing this is to introduce some additional local time derivative terms such as $(\overline{\rho' C'})_t$ that drop out in steady state, and can be shown to be small in unsteady situations). However, it is not possible to interpret the velocity variable in the conservation equations of present ocean models as being the density-weighted velocity, $\overline{\mathbf{u}}^{\rho}$, and arrive at the conclusions we have done here regarding the accuracy of Boussinesq ocean models. This is because the continuity equation of present ocean models does not carry $\overline{\rho}$ inside the divergence term. To attempt this interpretation leads to errors of the full Boussinesq magnitude (see Lu(2001)), which as Davis (1994) pointed out, can be as large as the effect of diapycnal mixing on the tracer equation.

In a recent article, Dukowicz (2001) has introduced a stiffer (less compressible) equation of state which has the effect of enabling a relatively accurate evaluation of the pressure gradient term, $\rho^{-1}\nabla_H p$, in the horizontal momentum equation. Paradoxically, this will lead to the entrenchment of the full Boussinesq conundrum error because there is then no choice but to interpret the horizontal velocity vector in the horizontal momentum equations as the Eulerian-mean horizontal velocity; it cannot be interpreted as the horizontal mass flux per unit area. This is because, while the pressure gradient term in the standard Boussinesq model, $\rho_o^{-1}\nabla_H p$, looks as though it suffers from the standard Boussinesq error, our reinterpretation of the model's velocity as the horizontal mass flux per unit area shows that, in fact, it is without error (see equations (26) and (29)). The implication is that modifying this term actually introduces error. Indeed, once we are forced to interpret the model's horizontal velocity as the horizontal Eulerian mean velocity, the non-divergence condition on the model's three-dimensional velocity vector then ensures that the model's velocity will be $\hat{\mathbf{u}}$ in the terminology of Section 3. The conservation equations (15) and (43) therefore apply, except that the models do not contain the source terms $\delta w\ \overline{C}_z$ whose neglect is associated with the Boussinesq conundrum. A separate achievement of the Dukowicz (2001) approach was a reduction in the transport errors identified by Dewar et al.(1998) due to models using an equation of state that is a function of height rather than of pressure. Recently, Griffies et al.(2000a) have shown that this issue can be overcome by simply using the pressure in the call to the equation of state from the previous time step of the model and this is now the default option in the MOM code.

The accurate conservation equations that need to be carried by a fully non-Boussinesq ocean model are given by (24)-(26). There are only four locations where the factor $\overline{\rho}/\rho_o$ (or its reciprocal) needs to be added to the Boussinesq conservation equations (27)-(29) to make them fully non-Boussinesq, and the most important of these is in the continuity equation, (24). Greatbatch et al. (2001) have modified the code of an existing Boussinesq ocean model to make it fully non-Boussinesq, and integrate the hydrostatic version of (24)-(26), and they describe the detailed changes that are needed to the model numerics as well as some results that illustrate the benefits of having a fully non-Boussinesq model. In particular, with a non-Boussinesq model it is possible to make direct comparison between the sea surface height and/or bottom pressure computed by the model, without the need to correct, as with a Boussinesq model, for the fact the model conserves volume rather than mass (e.g. Greatbatch(1994)). Likewise, concern over the averaged tracer equations, such as raised by MG, is automatically eliminated. However, it is important to appreciate that the Boussinesq/non-Boussinesq model intercomparisons shown in that paper are not able to throw light on the Boussinesq conundrum addressed here. This is because the error associated with the Boussinesq conundrum is of the same order of magnitude as the diapycnal mixing term, implying, in turn, that great care would be required to ensure that there is no spurious diapycnal mixing arising from the model numerics (Griffies et al., 2000b). Since a fully eddy-resolving calculation would be required (because the Boussinesq conundrum applies to averaged equations), demonstrating the Boussinesq conundrum is a particularly stringent test of a model (Griffies et al., 2000b), beyond the scope of the relatively simple model experiments described in Greatbatch et al.(2001).

Finally, we believe that the conundrum raised by MG and Davis(1994) points to the conclusion that the Boussinesq approximation consists of three parts, not two, as traditionally assumed. In addition to replacing (i) the equation for conservation of mass by the equation for conservation of volume and (ii) the density that appears in the temporal and advection operators by a constant reference density, it is important to also consider (iii) the error in the tracer equation resulting from using a divergence free velocity as the advecting velocity. As our analysis following equation (15) points out, it is this error that is at the heart of the conundrum raised by MG and Davis(1994). As far as we are aware, this third part to the Boussinesq approximation has not previously been pointed out explicitly, although it is implicit in the work of MG and Davis.

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Appendix

The average of the molecular diffusion term

The molecular diffusion in the tracer equation, (8), can be written

$$d_C = \rho^{-1}\nabla \cdot (\rho\kappa_C \nabla C) = \nabla \cdot (\kappa_C \nabla C) + \kappa_C \nabla C \cdot (\rho^{-1}\nabla\rho) \tag{44}$$

and if we consider the tracer potential temperature, then the last term in (44) scales as

$$\kappa_\theta \nabla\theta \cdot (\rho^{-1}\nabla\rho) = -\kappa_\theta \alpha \nabla\theta \cdot \nabla\theta. \tag{45}$$

When averaged, this becomes

$$-\kappa_\theta \overline{\alpha}\,\overline{\nabla\theta' \cdot \nabla\theta'} - \frac{\partial\alpha}{\partial\theta}\kappa_\theta \nabla\overline{\theta} \cdot \nabla(\overline{\theta'^2}) - \kappa_\theta \overline{\alpha} \nabla\overline{\theta} \cdot \nabla\overline{\theta}. \tag{46}$$

The first term here scales as $-\frac{1}{2}\overline{\alpha}\,\overline{\mathbf{u}'\theta'} \cdot \nabla\overline{\theta}$ which is smaller than the turbulent flux divergence term in (11), $-\nabla \cdot (\overline{\mathbf{u}'\theta'})$, by three orders of magnitude. Taking $\nabla(\overline{\theta'^2})$ to be given by the square of 3 degrees C over a distance of 100m, the second term in (46) is five orders of magnitude less that the dominant terms in (11). The third term in (46) is also five orders of magnitude less than the dominant terms in (11). We conclude, in agreement with McDougall and Garrett (1992), that to an excellent approximation,

$$\overline{d_C} = \nabla \cdot (\kappa_C \nabla \overline{C}) \tag{47}$$

and further, in (11), can be absorbed into the turbulent mixing term without incurring significant error.

The magnitude of $\overline{C'\nabla \cdot \mathbf{u}'}$

Using the functional form for the equation of state, $\rho = \rho(S, \theta, p)$, to find expressions for $\rho^{-1}\rho_t$ and $\rho^{-1}\nabla\rho$ in terms of the gradients of salinity, potential temperature and pressure, and using (7) and two versions of (8) (one for salinity and one for potential temperature), an expression for the instantaneous velocity divergence, $\nabla \cdot \mathbf{u}$, is obtained which we can use to find

$$\overline{C'\nabla \cdot \mathbf{u}'} = g\ \overline{c_s}^{-2}\,\overline{w'C'} + \overline{\alpha}\kappa_\theta \overline{C'\nabla \cdot (\nabla\theta')} - \overline{\beta}\kappa_S \overline{C'\nabla \cdot (\nabla S')}. \tag{48}$$

Here c_s is the speed of sound in seawater. The first term here can be estimated (say for the tracer potential temperature) using an eddy diffusivity of 10^{-5} m^2 s^{-1} operating on a vertical potential temperature gradient of 10^{-2} K m^{-1} giving $g\,\overline{c_s}^{-2}\overline{w'C'} = 10^{-12}$ K s^{-1} which is two orders of magnitude smaller than the diapycnal advection and diffusion terms in the conservation equation of potential temperature. The second term and third terms in (48) can be estimated by noting (again for the tracer potential temperature) that $\overline{\alpha}\kappa_\theta\overline{\theta'\nabla\cdot(\nabla\theta')} = \overline{\alpha}\kappa_\theta\overline{\nabla\cdot(\theta'\nabla\theta')} - \overline{\alpha}\kappa_\theta\overline{\nabla\theta'\cdot\nabla\theta'}$ and the divergence term can be ignored. The remaining term scales as $\frac{1}{2}\overline{\alpha}\overline{\mathbf{u}'\theta'}\cdot\nabla\overline{\theta}$ and this is no more than 10^{-13} K s^{-1} which is three orders of magnitude less than the effects of diapycnal mixing in the potential temperature equation. See Davis (1994) for further discussion of this $\overline{C'\nabla\cdot\mathbf{u}'}$ term.

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