MARINE MAMMAL SCIENCE, XX(X): On-line supplemental material (XXX 200X)

Appendix for "Vessel collisions with whales: the probability of lethal injury based on vessel speed"

One-dimensional collisions within the limits of the elastic and inelastic extremes

<u>Nomenclature:</u>
In all equations below, subscript 1 refers the vessel and subscript 2 refers to the whale. The prime indicates the respective post-collision momenta and velocities. The delta (Δ) indicates the change in either momentum ($\Delta\boldsymbol{p}$) or time (Δt), and boldface indicates vector quantities.

<u>Force and Momentum:</u>
Newton's Second Law, typically written as $\boldsymbol{F} = m\boldsymbol{a}$, where $\boldsymbol{a}$ is acceleration (m s^{-2}), states that the external force ($\boldsymbol{F}$, kg m s^{-2}) acting on a body with mass (m, kg) is equal to the rate of change in momentum ($\boldsymbol{p}$, kg m s^{-1}) of the body where momentum is the product of the mass and velocity ($\boldsymbol{v}$, m s^{-1}): $\boldsymbol{p} = m\boldsymbol{v}$.

Newton's Second Law in terms of momentum is:

$$\frac{d\boldsymbol{p}}{dt} = \frac{d(m\boldsymbol{v})}{dt} = m\frac{d\boldsymbol{v}}{dt} = m\boldsymbol{a} \qquad \text{Eq. 1}$$

and thus:

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} \doteq \frac{\Delta\boldsymbol{p}}{\Delta t}. \qquad \text{Eq. 2}$$

<u>Conservation of Linear Momentum:</u>
In all collisions, elastic or inelastic, the momentum of the system is conserved; i.e. when no net external force acts on a system the total linear momentum of the system cannot change – the total momentum of the system remains constant in magnitude and direction (i.e. $\boldsymbol{p}' = \boldsymbol{p}$) that can be written as:

$$m_1\boldsymbol{v}_1' + m_2\boldsymbol{v}_2' = m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2. \qquad \text{Eq. 3}$$

Conservation of momentum in Eq. 3 for one dimension (1D) provides one equation with the two unknown post-collision velocities (v_1' and v_2'). We can use a second equation from energy considerations to solve for the unknowns in this 1D elastic collision case. Thus, all further development below is 1D.

1D Elastic Collision: An elastic collision is one where the post-collision kinetic energy of the system is equal to the pre-collision kinetic energy of the system ($E'_{k1} + E'_{k2} = E_{k1} + E_{k2}$) that can be written as:

$$\frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \ . \qquad \text{Eq. 4}$$

We can rearrange Eq. 4 to:

$$m_2 \left(v_2'^2 - v_2^2 \right) = m_1 \left(v_1^2 - v_1'^2 \right)$$

and by expansion and rearrangement arrive at:

$$m_2 \left(v_2' - v_2 \right) \left(v_2' + v_2 \right) = m_1 \left(v_1 - v_1' \right) \left(v_1 + v_1' \right). \qquad \text{Eq. 5}$$

Rearranging Eq. 3 gives:

$$m_2 \left(v_2' - v_2 \right) = m_1 \left(v_1 - v_1' \right). \qquad \text{Eq. 6}$$

Dividing Eq. 5 by Eq. 6 yields:

$$v_2' + v_2 = v_1 + v_1' \qquad \text{Eq. 7}$$

and rearranging gives:

$$v_2' - v_1' = -\left(v_2 - v_1 \right). \qquad \text{Eq. 8}$$

Hence, for elastic collisions the relative speed of recession post-collision equals the relative speed of approach pre-collision.

Using the conservation of momentum (Eq. 3) and Eq. 8, the post-collision velocity of the whale is solved as follows:

$$m_1 v_1' + m_2 v_2' = m_1 v_1 + m_2 v_2,$$

and rearranging gives:

$$v_1' = \frac{m_1 v_1 + m_2 v_2 - m_2 v_2'}{m_1},$$

that upon substitution into Eq. 8 yields:

$$v_2' - \frac{m_1 v_1 + m_2 v_2 - m_2 v_2'}{m_1} = -\left(v_2 - v_1 \right).$$

Thus, by solving for v_2' one obtains:

$$v_2' = \frac{2 m_1 v_1 + m_2 v_2 - m_1 v_2}{m_1 + m_2}. \qquad \text{Eq. 9}$$

Substitution of the post-collision velocity (Eq. 9) into the momentum term in Eq. 2 gives:

$$F \doteq \frac{\Delta p}{\Delta t} \doteq \frac{p_2' - p_2}{\Delta t} \doteq \frac{m_2\left(\dfrac{2m_1v_1 + m_2v_2 - m_1v_2}{m_1 + m_2}\right) - m_2v_2}{\Delta t}. \qquad \text{Eq. 10}$$

The simplification of the Force Eq. 10 as follows:

$$F \doteq \frac{m_2}{\Delta t}\left[\frac{2m_1v_1 + m_2v_2 - m_1v_2}{m_1 + m_2} - v_2\right]$$

$$F \doteq \frac{m_2}{\Delta t}\left[\frac{2m_1v_1 + m_2v_2 - m_1v_2 - m_1v_2 - m_2v_2}{m_1 + m_2}\right]$$

$$F \doteq \frac{m_2}{\Delta t}\left[\frac{2m_1v_1 - 2m_1v_2}{m_1\left(1 + \dfrac{m_2}{m_1}\right)}\right]$$

$$F \doteq \frac{2m_2}{\Delta t}\left[\frac{m_1v_1\left(1 - \dfrac{v_2}{v_1}\right)}{m_1\left(1 + \dfrac{m_2}{m_1}\right)}\right],$$

that yields:

$$F \doteq \frac{2m_2v_1}{\Delta t}\left[\frac{1 - \dfrac{v_2}{v_1}}{1 + \dfrac{m_2}{m_1}}\right].$$

By approximation:

$$F \approx \frac{2m_2}{\Delta t}v_1 \text{ if } \frac{m_2}{m_1} \text{ and } \frac{v_2}{v_1} << 1,$$

then:

$$\left(1 - \frac{v_2}{v_1}\right) \approx 1 \text{ and } \left(1 + \frac{m_2}{m_1}\right) \approx 1,$$

and:

$$F \approx \frac{2m_2}{\Delta t}v_1.$$

Thus, the forces involved in the elastic collision are the product of the mass of the whale and the speed of the vessel.

<u>1D Inelastic Collision:</u> A perfectly inelastic collision is one where only the momentum of the system is conserved and the post-collision velocities of the two colliding bodies are equal and move as one body at velocity v' (i.e. $v' = v_1' = v_2'$). By using Eq. 3 the post-collision velocity is defined as:

$$v' = \frac{m_1 v_1 + m_2 v_2}{(m_1 + m_2)}. \qquad \text{Eq. 11}$$

The substitution of the post-collision velocity (Eq. 11) into the momentum term in Eq. 2 gives:

$$F \doteq \frac{\Delta p}{\Delta t} \doteq \frac{p_2' - p_2}{\Delta t} \doteq \frac{m_2\left(\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}\right) - m_2 v_2}{\Delta t}. \qquad \text{Eq. 12}$$

The simplification of the Force Eq. 12 as follows:

$$F \doteq \frac{m_2}{\Delta t}\left[\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} - v_2\right]$$

$$F \doteq \frac{m_2}{\Delta t}\left[\frac{m_1 v_1 + m_2 v_2 - m_1 v_2 - m_2 v_2}{m_1\left(1 + \frac{m_2}{m_1}\right)}\right]$$

$$F \doteq \frac{m_2}{\Delta t}\left[\frac{m_1 v_1\left(1 - \frac{v_2}{v_1}\right)}{m_1\left(1 + \frac{m_2}{m_1}\right)}\right],$$

that yields:

$$F \doteq \frac{m_2 v_1}{\Delta t}\left[\frac{1 - \frac{v_2}{v_1}}{1 + \frac{m_2}{m_1}}\right].$$

By approximation:

$$F \approx \frac{m_2}{\Delta t} v_1 \text{ if } \frac{m_2}{m_1} \text{ and } \frac{v_2}{v_1} << 1,$$

then:

$$\left(1 - \frac{v_2}{v_1}\right) \approx 1 \text{ and } \left(1 + \frac{m_2}{m_1}\right) \approx 1,$$

and

$$F \approx \frac{m_2}{\Delta t} v_1.$$

Thus, the forces involved in the perfectly inelastic collision are the product of the mass of the whale and the speed of the vessel.